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Multiplier sequences, classes of generalized Bessel functions and open problems

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ABSTRACT

Motivated by the study of the distribution of zeros of generalized Bessel-type functions, the principal goal of this paper is to identify new research directions in the theory of multiplier sequences. The investigations focus on multiplier sequences interpolated by functions which are not entire and sums, averages and parametrized families of multiplier sequences. The main results include (i) the development of a 'logarithmic' multiplier sequence and (ii) several integral representations of a generalized Bessel-type function utilizing some ideas of G.H. Hardy and L.V. Ostrovskii. The explorations and analysis, augmented throughout the paper by a plethora of examples, led to a number of conjectures and intriguing open problems.

1. Introduction

In 1905 G.H. Hardy [13] studied the following entire functions of exponential type, as generalizations of e^z:

$$E_{s,a}(z) := \sum_{n=0}^{\infty} (n+a)^s \frac{z^n}{n!}, \qquad s \in \mathbb{R}, \quad a \geq 0. \tag{1.1}$$

Although Hardy allowed the parameters to be complex numbers, in the present paper we will only consider parameters satisfying the restrictions in (1.1). Note that $E_{0,a} = e^z$, and for $k \in \mathbb{N}$, $E_{k,a} = e^z T_k(z)$, where $T_k(z)$ is a polynomial of degree k. If $a = 0$, we set $E_{s,0} = \sum_{n=1}^{\infty} n^s \frac{x^n}{n!}$, $s \in \mathbb{R}$. In [18], I.V. Ostrovskii describes the real zeros of these generalized exponential functions.

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Theorem 1. *(See [18, Theorem 2.5].) Let $E_{s,a}$ be defined as in (1.1), and let $k \in \mathbb{N}_0$.*

(a) *For $k < s < k+1$, $E_{s,a}$ has only $k+1$ real zeros.*
(b) *For $s < 0$, $E_{s,a}$ does not have any real zeros.*

A consequence of Theorem 1 is that $\{(k+a)^s\}_{k=0}^{\infty}$ is not a multiplier sequence (cf. Definition 4) for non-integral or negative s. The observation that the sequence $\{1/k!\}_{k=0}^{\infty}$ is a complex zero decreasing sequence (cf. Definition 3), however, motivates the study of functions of the form

$$B_{s,a}(z) := \sum_{n=0}^{\infty}(n+a)^s \frac{z^n}{n!n!} \tag{1.2}$$

along with the location of their zeros (see Section 3). We close this section with some definitions, and the general question which led to most of the work and considerations in this paper.

Definition 1. A real entire function $\varphi(x) = \displaystyle\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is said to belong to the *Laguerre–Pólya class*, written $\varphi \in \mathscr{L} - \mathscr{P}$, if it admits the representation

$$\varphi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\omega}\left(1+\frac{x}{x_k}\right)e^{-x/x_k},$$

where $b, c \in \mathbb{R}$, $x_k \in \mathbb{R}\setminus\{0\}$, m is a non-negative integer, $a \geq 0$, $0 \leq \omega \leq \infty$ and $\displaystyle\sum_{k=1}^{\omega}\frac{1}{x_k^2} < +\infty$.

Definition 2. A real entire function $\varphi(x) = \displaystyle\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ is said to be of *type I* in the Laguerre–Pólya class, written $\varphi \in \mathscr{L} - \mathscr{P}I$, if $\varphi(x)$ or $\varphi(-x)$ admits the representation

$$\varphi(x) = cx^m e^{\sigma x} \prod_{k=1}^{\omega}\left(1+\frac{x}{x_k}\right),$$

where $c \in \mathbb{R}$, m is a non-negative integer, $\sigma \geq 0$, $x_k > 0$, $0 \leq \omega \leq \infty$ and $\displaystyle\sum_{k=1}^{\omega}\frac{1}{x_k} < +\infty$. If $\gamma_k \geq 0$ for $k = 0, 1, 2, \ldots$, we write $\varphi \in \mathscr{L} - \mathscr{P}^+$. Finally, $\mathscr{L} - \mathscr{P}(-\infty, 0]$ denotes the class of functions in $\mathscr{L} - \mathscr{P}$ whose zeros lie in $(-\infty, 0]$.

We point out that a real entire function φ belongs to $\mathscr{L} - \mathscr{P}I$ if and only if its Taylor coefficients are of the same sign, or alternate in sign. Thus $\mathscr{L} - \mathscr{P}^+ \subset \mathscr{L} - \mathscr{P}I \subset \mathscr{L} - \mathscr{P}$.

Definition 3. A sequence of real numbers $\{\gamma_k\}_{k=0}^{\infty}$ is called a *complex zero decreasing sequence, or CZDS*, if the linear operator T defined by $T[x^k] = \gamma_k x^k$ has the property that for every real polynomial $p(x)$,

$$Z_C(T[p(x)]) \leq Z_C(p(x)),$$

where $Z_C(p)$ denotes the number of non-real zeros of the polynomial p, counting multiplicity.