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Tail variational principle for a countable discrete amenable group action

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ABSTRACT

In this paper, we define the measure-theoretic tail entropy for a countable discrete amenable group action and prove the tail variational principle when there exists a refining sequence of essential partitions. Additionally, some properties of local entropy are investigated.

1. Introduction

Topological tail entropy quantifies the complexity of a dynamical system at arbitrarily small scales. It captures the entropy near any single orbit. This quantity was first introduced for dynamics of $\mathbb{Z}$-action by Misiurewicz in [15] and was deeply studied by many others (e.g., see [2,3,5,13,23]). (Historically, Misiurewicz and Buzzi called it topological conditional entropy and local entropy respectively.) It is well known that the variational principle plays a fundamental role in ergodic theory and dynamical systems. In the case of $\mathbb{Z}$-action, Ledrappier [11] obtained a variational principle of topological tail entropy and Downarowicz [5] established a variational principle between topological tail entropy and entropy structure. Later, Burguet [2] presented a direct proof of Downarowicz's results and extended them to a noninvertible case. Recently, Zhou–Zhang–Chen [23] extended Ledrappier's variational principle to the case of amenable group action. That is, they investigated the topological tail entropy for the actions of countable discrete amenable groups and established the variational principle for it. Moreover, Chung–Zhang [4] also defined topological tail entropy for actions of sofic groups in the same spirit of Misiurewicz's ideas and they investigated the asymptotical h-expansiveness for actions of amenable groups along the line of tail entropy.

In this paper, we introduce the measure-theoretic tail entropy for a countable discrete amenable group action and establish the variational principle between topological tail entropy and measure-theoretic tail

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entropy when there exists a refining sequence of essential partitions. This is Theorem 2.2. We note that even for $\mathbb{Z}$-actions, our definition of measure-theoretic tail entropy is new. We also obtained a variational principle for a function of measures which is defined by refining sequence of essential partitions. At the same time, some properties of local entropy are investigated in Section 4. Let us also note that except Proposition 4.4 which uses Quasitiling Theorem, many other statements are generalizations of results for $\mathbb{Z}$-actions.

2. Main results

A G-system (X, G) means that (X, d) is a compact metric space and G is a countable discrete amenable group which acts on X. We refer the readers to [9,16,19] for entropy theory of amenable group actions. Denote by $\mathcal{M}(X, G)$ the set of G-invariant Borel probability measures. It is compact with respect to the weak-star topology.

We firstly recall the definition of topological tail entropy.

For a non-empty set $Y \subset X$, an open cover $\mathcal{U}$ of X and a finite set $F \subset G$, write

$$N(\mathcal{U}, Y) = \min\{\#\mathcal{V} : \ \mathcal{V} \subset \mathcal{U}, Y \subset \bigcup_{V \in \mathcal{V}} V\}$$

and

$$\mathcal{U}^F = \bigvee_{g \in F} g^{-1}\mathcal{U}.$$

Given two open covers $\mathcal{U}$ and $\mathcal{V}$ of X with finite elements, define

$$N(\mathcal{U}|\mathcal{V}) = \max_{V \in \mathcal{V}} N(\mathcal{U}, V),$$

$$h(G, \mathcal{U}|\mathcal{V}) = \lim_{F} \frac{1}{|F|} \ln N(\mathcal{U}^F|\mathcal{V}^F),$$

$$h(G|\mathcal{V}) = \sup_{\mathcal{U}} h(G, \mathcal{U}|\mathcal{V}),$$

$$h^*(X, G) = \inf_{\mathcal{V}} h(G|\mathcal{V}),$$

here $|F|$ is the number of elements of F and the limit $\lim_F$ is defined at the beginning of Section 3. We call $h^*(X, G)$ the *topological tail entropy*. Let us note that the limit exists in the definition of $h(G, \mathcal{U}|\mathcal{V})$ (see Lemma 3.3 in Section 3).

For a partition ξ of X and a finite set $F \subset G$, we write

$$\xi^F = \bigvee_{g \in F} g^{-1}\xi.$$

Recall that for a probability measure μ on X and two finite measurable partitions $\xi = \{A_i\}_i$ and $\eta = \{B_j\}_j$ of X,

$$H_\mu(\xi|\eta) = -\sum_{i,j} \mu(A_i \cap B_j) \ln \frac{\mu(A_i \cap B_j)}{\mu(B_j)} = \sum_j \mu(B_j) H_{\mu_{B_j}}(\xi|B_j),$$

here μ_{B_j} is the conditional measure of μ on B_j.

In the following, by a partition we always mean a finite partition of X by Borelian sets.