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Eventually positive semigroups of linear operators

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Dedicated with great pleasure to Wolfgang Arendt on the occasion of his 65th birthday

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ABSTRACT

We develop a systematic theory of eventually positive semigroups of linear operators mainly on spaces of continuous functions. By eventually positive we mean that for every positive initial condition the solution to the corresponding Cauchy problem is positive for large enough time. Characterisations of such semigroups are given by means of resolvent properties of the generator and Perron–Frobenius type spectral conditions. We apply these characterisations to prove eventual positivity of several examples of semigroups including some generated by fourth order elliptic operators and a delay differential equation. We also consider eventually positive semigroups on arbitrary Banach lattices and establish several results for their spectral bound which were previously only known for positive semigroups.

1. Introduction

One of the distinguishing features of many second-order parabolic boundary value problems is their positivity preserving property: if the initial condition is positive, so is the solution at all positive times. Such equations are frequently expressed as an abstract Cauchy problem of the form

$$\dot{u}(t) = Au(t) \quad \text{if } t \geq 0, \qquad u(0) = u_0, \tag{1.1}$$

on a (complex) Banach lattice E such as $L^p(\Omega)$ or $C(\overline{\Omega})$, where A is the generator of a strongly continuous semigroup. If we represent the solution of (1.1) in terms of the corresponding semigroup $(e^{tA})_{t\geq 0}$, then

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positivity means that $u_0 \geq 0$ implies $e^{tA}u_0 \geq 0$ for all $t \geq 0$. There is a sophisticated general theory of positive semigroups, which has found a large number of applications; see for instance [3].

However, if A is the realisation of a *differential* operator, such positivity—which is usually obtained as a consequence of the maximum principle—is surprisingly rare. Indeed, under mild auxiliary assumptions on the operator A, *a priori* of arbitrary order, positivity of the semigroup $(e^{tA})_{t\geq 0}$ already implies that A is second-order elliptic if $E = L^p(\mathbb{R}^d)$ [24] or $E = C(\overline{\Omega})$ [6].

In such a case, in an attempt to bypass this restriction, we could weaken the requirement on the semigroup and stipulate that $e^{tA}u_0$ merely be positive for $t \geq 0$ "large enough" whenever $u_0 \geq 0$. Indeed, in recent times various disparate examples of such "eventually positive semigroups" have emerged, all seemingly completely independent of each other. Here are some examples, many of which we will consider in more detail below, in Section 6.

A matrix exponential e^{tA} can be positive for large t even if A has some negative off-diagonal entries, a phenomenon which seems to have been observed only quite recently; see [26] and the references therein.

For elliptic operators of order $2m$, $m \geq 2$, there is no maximum principle in general. The resolvent of the bi-Laplacian exhibits positivity properties on very few domains such as balls and perturbations of balls; see [10,19]. The question as to whether the corresponding parabolic problem becomes "essentially" positive for large $t > 0$ was investigated in [15,17].

Another example is the *Dirichlet-to-Neumann operator* D_λ associated with $\Delta u + \lambda u = 0$ on a domain Ω. For λ on one side of the first Dirichlet eigenvalue, the semigroup generated by $-D_\lambda$ on $L^2(\partial\Omega)$ is positive as shown in [7]. For other values of λ the semigroup may be positive, eventually positive or neither as the example of the disc shows; see [12]. The present paper had its genesis in an attempt to understand this phenomenon better at a theoretical level. We provide a detailed discussion in Section 6.2.

A further example is provided by certain delay differential equations. Under special assumptions they generate positive semigroups; see [13, Theorem VI.6.11]. We will show in Section 6.5 that there are also situations where the semigroup is eventually positive without being positive.

The variety of examples suggests that eventually positive semigroups could prove more ubiquitous than their positive counterparts, and no doubt more examples will emerge. Quite surprisingly, to date there seems to have been no unified treatment of such objects, in marked contrast to the positive case. Here, and in an envisaged sequel [11], we intend to address this. Our abstract theory will allow us to recover several known results and to prove some new ones in the above-mentioned areas.

Our main focus in this article is the investigation of strongly continuous semigroups with eventual positivity properties on $C(K)$, the space of complex-valued continuous functions on a compact non-empty Hausdorff space K. In order to give an idea of our results, we first need to introduce some notation. We call f *positive* if $f(x) \geq 0$ for all $x \in K$ and write $f \geq 0$. If $f \geq 0$ but $f \neq 0$ we write $f > 0$; we call f *strongly positive* and write $f \gg 0$ if there exists $\beta > 0$ such that $f \geq \beta\mathbf{1}$, where $\mathbf{1}$ is the constant function on K with value one. A bounded linear operator T on $C(K)$ is called *strongly positive*, denoted by $T \gg 0$, if $Tf \gg 0$ whenever $f > 0$, and similarly, a linear functional $\varphi : C(K) \to \mathbb{C}$ is called *strongly positive*, again denoted by $\varphi \gg 0$, if $\varphi(f) > 0$ for each $f > 0$.

If $A\colon D(A) \to E$ is a closed operator with domain $D(A)$ on the Banach space E we denote by $\sigma(A)$ and $\varrho(A) := \mathbb{C} \setminus \sigma(A)$ the *spectrum* and *resolvent set* of A, respectively. Any point in $\sigma(A)$ is called a *spectral value*. We call

$$\mathrm{s}(A) := \sup\{\operatorname{Re}\lambda\colon \lambda \in \sigma(A)\} \in [-\infty, \infty] \tag{1.2}$$

the *spectral bound* of A. For some classes of positive semigroups the spectral bound $\mathrm{s}(A)$ is necessarily a *dominant spectral value*, by which we mean that $\mathrm{s}(A) \in \sigma(A)$ and that the *peripheral spectrum*

$$\sigma_{\mathrm{per}}(A) := \sigma(A) \cap (\mathrm{s}(A) + i\mathbb{R}) \tag{1.3}$$