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Embeddability of homeomorphisms of the circle in set-valued iteration groups

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ABSTRACT

Let $F: \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism without periodic points. It is known that F is embeddable in a continuous iteration group if and only if F is minimal. We deal with F which is not minimal. In this case, F satisfying some additional assumptions can be embedded but only in a nonmeasurable iteration groups. There are infinitely many such nonmeasurable groups. We propose here a new approach to the problem of embeddability. For a given homeomorphism F without periodic points we construct some substitute of an iteration group, namely the unique special set-valued iteration group $\{F^t: \mathbb{S}^1 \to \mathrm{cc}[\mathbb{S}^1], t \in \mathbb{R}\}$, which is regular in a sense and in which F can be embedded i.e. $F(x) \in F^1(x)$. We also determine a maximal countable and dense subgroup $T \subset \mathbb{R}$ such that $\{F^t: \mathbb{S}^1 \to \mathrm{cc}[\mathbb{S}^1], t \in T\}$ has a continuous selection $\{f^t: \mathbb{S}^1 \to \mathbb{S}^1, t \in T\}$ being the best regular embedding of F. If there exists a nonmeasurable embedding $\{f^t: \mathbb{S}^1 \to \mathbb{S}^1, t \in \mathbb{R}\}$ of F, then there exists an additive function $\gamma: \mathbb{R} \to T$ such that $f^t(z) \in F^{\gamma(t)}(z), t \in \mathbb{R}$. We determine a unique maximal subgroup T with this property.

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1. Introduction

Notions and basic facts.

Let $\mathbb{S}^1 := \{z \in \mathbb{C}, |z| = 1\}$ be the unit circle with positive orientation and $\mathrm{cc}[\mathbb{S}^1]$ be the family of all nonempty convex and compact subsets of $\mathbb{S}^1$, that is the family of closed arcs and points of $\mathbb{S}^1$. Let $F: \mathbb{S}^1 \to \mathbb{S}^1$ be a homeomorphism without periodic points, which is equivalent to the fact that its rotation number ϱ_F is irrational.

A family $\{f^t, t \in \mathbb{R}\}$ of homeomorphisms $f^t: \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$f^t \circ f^s = f^{t+s}, \quad t, s \in \mathbb{R}$$

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is said to be an *iteration group* or a *flow*. If for every $z \in \mathbb{S}^1$ the mapping $t \to f^t(z)$ is continuous then iteration group is said to be *continuous*. A homeomorphism F is said to be *(continuously) embeddable* if there exists a (continuous) iteration group $\{f^t, t \in \mathbb{R}\}$ such that $f^1 = F$. This iteration group is said to be *the embedding of* F.

It is known that the set $L_F := \{F^n(z), n \in \mathbb{Z}\}^d$, the set of limit points of orbits of F, does not depend on the choice of $z \in \mathbb{S}^1$ and L_F either equals $\mathbb{S}^1$ or L_F is a nowhere dense perfect set (see e.g. [6]). A homeomorphism F is continuously embeddable if and only if F is minimal, that is when $L_F = \mathbb{S}^1$. Then continuous embedding is unique up to a constant (see [13]). We deal with the case $L_F \neq \mathbb{S}^1$. In such a case F may possess embeddings but nonmeasurable with respect to the time parameter t. K. Ciepliński in [2] gave the necessary and sufficient condition for embeddability in the discontinuous iteration groups. In this case F has infinitely many nonmeasurable embeddings.

For a given homeomorphism without periodic points we construct some substitution of an iteration group, namely the special set-valued iteration group $\{F^t: \mathbb{S}^1 \to \mathrm{cc}[\mathbb{S}^1], t \in \mathbb{R}\}$ and a countable and dense subgroup $T \subset \mathbb{R}$ such that $\{F^t: \mathbb{S}^1 \to \mathrm{cc}[\mathbb{S}^1], t \in T\}$ has a continuous selection $\{f^t: \mathbb{S}^1 \to \mathbb{S}^1, t \in T\}$ being an embedding of F restricted to T possessing several regular properties. We determine a unique maximal subgroup T with the above mentioned properties. The idea of generalized embedding called phantom iterates has been presented and widespread by Gy. Targonski (see [10,11]). He considered another approach using the semigroups of Koopman operators. In this note we apply the set-valued iteration semigroups. These semigroups were introduced and studied in a very general case by A. Smajdor in [9] (see also [7,8,15]).

2. Preliminaries

Let $\Pi(t) := (e^{2\pi i t})_{|[0,1)}$. If $u, w, z \in \mathbb{S}^1$, then there exist unique $t_1, t_2 \in [0,1)$ such that $w\Pi(t_1) = z$ and $w\Pi(t_2) = u$. Introduce a cyclic order $\prec$ on the circle (see [1])

$$u \prec w \prec z \quad \text{if and only if} \quad 0 < t_1 < t_2$$

and

$$u \preceq w \preceq z \quad \text{if and only if} \quad t_1 \leq t_2 \quad \text{or} \quad t_2 = 0.$$

We say that $\varphi: D \subset \mathbb{S}^1 \to \mathbb{S}^1$ is *increasing (strictly increasing)* if for every $u, w, z \in \mathbb{S}^1$ such that $u \prec w \prec z$ we have $\varphi(u) \preceq \varphi(w) \preceq \varphi(z)$ (resp. $\varphi(u) \prec \varphi(w) \prec \varphi(z)$).

Note that the rotation functions $R(z) = az$ for $a \in \mathbb{S}^1$ are strictly increasing.

The following generalization of result of Poincare on the solutions of the Schröder equation

$$\Phi(F(z)) = e^{2\pi i \varrho}\Phi(z), \qquad z \in \mathbb{S}^1, \tag{1}$$

where ϱ is the rotation number of homeomorphism F, will serve as one of the main tools of our investigation (see [5,12]).

Proposition 1. *If a homeomorphism $F: \mathbb{S}^1 \to \mathbb{S}^1$ has no periodic points then there exists exactly one continuous solution of equation (1) such that $\Phi(1) = 1$. This solution is surjective and increasing. Moreover, Φ is a homeomorphism if and only if $L_F = \mathbb{S}^1$.*

From now $L_F \neq \mathbb{S}^1$. Since L_F is a nowhere dense and perfect set $\mathbb{S}^1 \setminus L_F$ is a countable sum of pairwise disjoint open arcs. Denote the family of these arcs by $\mathcal{A}$. Denote by $\alpha(I)$ the middle point of the arc $I \subsetneq \mathbb{S}^1$ and let $M := \{\alpha(I),\ I \in \mathcal{A}\}$. Put $I_p := \alpha^{-1}(p)$ for $p \in M$. Hence we have the decomposition

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