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Blow-up and global solutions for a class of nonlinear reaction diffusion equations under Dirichlet boundary conditions

Juntang Ding *, Hongjuan Hu

School of Mathematical Sciences, Shanxi University, Taiyuan 030006, PR China

A R T I C L E I N F O

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A B S T R A C T

In this paper, we consider the following reaction diffusion equations under Dirichlet boundary condition

$$\begin{cases} (g(u))_t = \nabla \cdot (\rho(|\nabla u|^2)\nabla u) + k(t)f(u) & \text{in } \Omega \times (0, t^*), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. By constructing some appropriate auxiliary functions and using a first-order differential inequality technique, the upper and lower bounds on blow-up time when blow-up occurs are presented. Moreover, conditions on the data to ensure that the solution exists globally or blows up at some finite time are also derived.

1. Introduction

During the past few years, a blow-up phenomenon of solutions for nonlinear reaction diffusion equations and systems has been extensively investigated as can be seen by viewing the list of references in [1–4,9,10,17–19] and the references therein. Further contributions to the field are [5–8,11–15,20,21]. These literatures studied the questions of global existence, blow-up at some finite time, blow-up rate, blow-up set, asymptotic behavior for solutions and so on as well as a variety of methods used to research these questions. Particularly, the problems of the blow-up and global solutions for nonlinear reaction diffusion equations under Dirichlet boundary conditions have been considered in [5,8,11–15]. In this paper we discuss the blow-up and global solutions for the following nonlinear reaction diffusion equations under Dirichlet boundary condition

* Corresponding author.
E-mail address: djuntang@sxu.edu.cn (J. Ding).

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$$\begin{cases} (g(u))_t = \nabla \cdot (\rho(|\nabla u|^2)\nabla u) + k(t)f(u) & \text{in } \Omega \times (0, t^*), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, $\overline{\Omega}$ is the closure of Ω, and ∇ is the gradient operator. Set $\mathbb{R}^+ = (0, +\infty)$. We assume that g is a $C^2(\mathbb{R}^+)$ function with $g'(s) > 0$ for all $s > 0$, ρ is a positive $C^1(\mathbb{R}^+)$ function which satisfies the ellipticity condition $\rho(s) + 2s\rho'(s) > 0$ for all $s > 0$, f is a nonnegative $C(\mathbb{R}^+)$ function, k is a positive $C^1(\mathbb{R}^+)$ function, u_0 is a nonnegative $C^1(\overline{\Omega})$ function satisfying the compatibility condition $u_0(x) = 0$ on $\partial\Omega$. It follows from the parabolic maximum principle [16] that the solution of (1.1) is nonnegative in $x \in \overline{\Omega}$ and $t \in [0, t^*)$.

The papers [11,13,14] have studied the special cases of problem (1.1). In paper [14], the following problem was considered by Payne and Schaefer

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \Omega \times (0, t^*), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. They used a differential inequality technique and a comparison principle to obtain a lower bound on blow-up time when blow-up occurs. Little later, Payne, Philippin and Schaefer investigated the problem in [13]

$$\begin{cases} u_t = \nabla \cdot (\rho(|\nabla u|^2)\nabla u) + f(u) & \text{in } \Omega \times (0, t^*), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. Under appropriate assumptions on the functions f, ρ and u_0, a lower bound on blow-up time was showed by applying a differential technique when blow-up does occur. Moreover, a criterion for blow-up and conditions which ensure that blow-up cannot occur were obtained. Finally, the following problem was studied by Payne and Philippin in [11]

$$\begin{cases} u_t = \Delta u + k(t)f(u) & \text{in } \Omega \times (0, t^*), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0, t^*), \\ u(x,0) = u_0(x) \geq 0 & \text{in } \overline{\Omega}, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$. A first-order differential inequality technique and Sobolev inequality were used to give the sufficient conditions which guarantee the blow-up or the global existence of the solution. In addition, lower and upper bounds on blow-up time were also derived.

In this paper, the more general problems (1.1) are studied. It seems that the auxiliary functions defined in [11,13,14] are not applicable for the problem (1.1). By defining completely different auxiliary functions from those in [11,13,14] and using a first-order differential inequality technique, we obtain conditions sufficient to ensure the solution exists for all time or blows up at some finite time. The upper and lower bounds on blow-up time are also given. Our results extend and supplement those obtained in [11,13,14].

We proceed as follows. In Section 2, we establish a sufficient condition on the data of problem (1.1) to guarantee the blow-up of the solution and obtain an upper bound on blow-up time. In Section 3, a lower bound on blow-up time when blow-up occurs is imposed. In Section 4, we derive conditions on the data of problem (1.1) sufficient to ensure the global existence of the solution and in Section 5, some examples are given to demonstrate the applications of abstract results.