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Nonlocal Cauchy problems close to an asymptotically stable equilibrium point

Adrian Viorel [a,b,*]

[a] Department of Mathematics, Technical University of Cluj Napoca, Str. Memorandumului 28, 400114 Cluj-Napoca, Romania
[b] Department of Mathematics, Babeş-Bolyai University Cluj-Napoca, Str. Kogalniceanu 3, 400084 Cluj-Napoca, Romania

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ABSTRACT

In this work we investigate the existence of solutions for semilinear Cauchy problems with nonlocal initial conditions in the neighborhood of an asymptotically stable equilibrium point of the evolution equation. Using Granas' continuation principle for contractive maps and the qualitative theory of differential equations in Banach spaces, under mild assumptions, we prove the existence of a unique solution. We also show that the main abstract result can be applied to nonlocal initial boundary value problems for reaction–diffusion equations with non-convex nonlinearities.

1. Introduction and preliminaries

In the present article, we consider nonlocal Cauchy problems associated to a semilinear evolution equation in a Banach space X

$$\begin{cases} \dfrac{du}{dt} + Au = f(u), \quad t \in [0,T], \\ u(0) = \int_0^T u(t)\, h(t)\, dt, \end{cases} \tag{1}$$

for a given weight function $h \in L^1(0,T;\mathbb{R}_+)$ and a sectorial operator $A : D(A) \subseteq X \to X$.

Nonlocal conditions (and nonlocal Cauchy problems) have been pioneered by L. Byszewski a couple of decades ago and we refer to [4,6] or [11] for a detailed discussion about the motivation and advantages of nononlocal problems over the classical initial value problems, which they extend.

* Correspondence to: Department of Mathematics, Technical University of Cluj Napoca, Str. Memorandumului 28, 400114 Cluj-Napoca, Romania.
E-mail address: Adrian.Viorel@math.utcluj.ro.

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To study the existence of solutions for semilinear nonlocal problems, the most common approach is to rewrite (1) as a fixed point problem using the semigroup e^{-tA} generated by A and the variation of constants formula which yields

$$u(t) = e^{-tA} \int_0^T u(s)\,h(s)\,ds + \int_0^t e^{-(t-s)A} f(u(s))\,ds.$$

Then, one can apply different fixed point principles to the above equation and establish the existence of mild solutions (see for example [1,2,4–7,10] or [14]).

We do not pursue this approach further, but rather, our aim is to show that if the evolution equation has an asymptotically stable equilibrium point u^*, then, under some mild assumptions, the nonlocal problem has a solution close to u^*.

The motivation behind this new approach is twofold. First, by a classical result for differential equations in Banach spaces (see [9] or [13]), it is known that the Cauchy problem

$$\begin{cases} \dfrac{du}{dt} + Au = f(u), & t \geq 0, \\ u(0) = u_0, \end{cases}$$

has a unique, global in time solution $u(t; u_0)$ for any initial datum $u_0 \in U$ in a sufficiently small neighborhood U of the asymptotically stable equilibrium point u^*. Hence, finding a solution to the nonlocal problem (1) is equivalent to finding an appropriate initial state $u_0 \in U$ such that the trajectory $u(t; u_0)$ originating from u_0 satisfies the nonlocal condition

$$u_0 = \int_0^T u(t; u_0)\,h(t)\,dt =: H(u_0).$$

This is a fixed point problem in $U \subset X$, for the nonlinear integral operator H.

Secondly, provided that

$$\int_0^T h(t)\,dt = 1, \tag{2}$$

one can easily observe that the constant function $u(t) \equiv u^*$ solves the nonlocal Cauchy problem (1) since

$$u^* = \int_0^T u^* h(t)\,dt.$$

In this context, our intention is to prove the existence of a solution to (1) near u^* by homotopically connecting the nonlocal problem with a given h to a problem whose weight function satisfies (2) and which is known to have a (constant) solution. In order to prove our main result, we will thus rely on Granas' well-known continuation principle for contractive maps.

Theorem 1.1. *(See Granas [8].) Let $(X, \|\cdot\|)$ be a Banach space and $(H_\lambda)_{\lambda\in[0,1]}$ a family of nonlinear operators $H_\lambda : \overline{U} \to X$, where $U \subseteq X$ is an open set in X. Assume that there exist $0 \leq L < 1$ and $K > 0$ such that*