J. Math. Anal. Appl. 433 (2016) 1743–1756

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Stable equilibria to a singularly perturbed reaction–diffusion equation in a degenerated heterogeneous environment

Arnaldo Simal do Nascimento [a,*], Maicon Sônego [b]

[a] *Universidade Federal de São Carlos – DM, 13565-905 S. Carlos, SP, Brazil*
[b] *Universidade Federal de Itajuba – IMC, 37500-903 Itajuba, MG, Brazil*

A R T I C L E I N F O

Article history:
Received 19 November 2014
Available online 2 September 2015
Submitted by Y. Yamada

Keywords:
Reaction–diffusion equations
Patterns
Stable equilibria
Γ-convergence
Degenerated heterogeneous environment

A B S T R A C T

We address the problem $u_t = \epsilon^2 \Delta u + f(u, x)$ in $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ under boundary condition $\partial_\nu u = 0$ where $f(u, x) = -(u - a(x))(u - \theta(x))(u - b(x))$, $\theta(x) = [a(x) + b(x)]/2$ and $a \leq b$ in Ω. The novelty here lies in the fact that the roots of f are allowed to degenerate in the sense that $a = \theta = b$ in $\Omega \setminus D$ where $D \subset \Omega$ is such that $D = D_1 \cup D_2$, $\overline{D_1} \cap \overline{D_2} = \emptyset$, D_1 and D_2 are non-empty open connected sets with Lipschitz-continuous boundaries and $a < b$ in D. For ϵ small, we prove existence of four families of stable stationary solutions u_ϵ approaching the roots of f in the topology of L^1. Our approach is variational and based on Γ-convergence theory.

1. Introduction and main result

The main concern in this paper is to prove existence of nonconstant stable stationary solutions – herein referred to as patterns, for short – to the reaction–diffusion problem

$$\left.\begin{array}{ll} u_t = \epsilon^2 \Delta u + f(u, x), & (t, x) \in \mathbb{R}^+ \times \Omega \\ \frac{\partial u}{\partial \nu} = 0, & (t, x) \in \mathbb{R}^+ \times \partial\Omega \end{array}\right\} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an open connected bounded set with C^2 boundary, ϵ is a small positive parameter, ν denotes the outer unit normal to $\partial\Omega$ and $f(u, x)$ is defined by

$$f(u, x) = -(u - a(x))(u - \theta(x))(u - b(x)) \tag{2}$$

where $\theta(x) = [a(x) + b(x)]/2$ and $a \leq b$ in Ω. The novelty here lies in the fact that the roots of f are allowed to degenerate in the sense described below.

* Corresponding author.
E-mail addresses: arnaldon@dm.ufscar.br (A.S. do Nascimento), mcn.sonego@unifei.edu.br (M. Sônego).

http://dx.doi.org/10.1016/j.jmaa.2015.08.071
0022-247X/© 2015 Published by Elsevier Inc.

Let $D \subset \Omega$ be such that $D = D_1 \cup D_2$, $\overline{D_1} \cap \overline{D_2} = \emptyset$, where D_1 and D_2 are open and connected sets with Lipschitz-continuous boundaries and set

$$D_3 \overset{\text{def}}{=} \Omega \setminus \overline{D}.$$

The roots of f are required to satisfy

(f_1) $a, \theta, b \in C(\Omega) \cap C^1(\cup_{i=1}^3 D_i)$, C^1-bounded in D_1, D_2 and D_3 and $a = \theta = b$ in D_3,
(f_2) $a < b$ in D.

Note that a and b may not be differentiable across $\Omega \cap \partial D_3$ and the sets D_1 and D_2 may be located in Ω in such way that one of the following situations may occur

- $\partial D_j \cap \partial\Omega$ $(j = 1, 2)$ are sets of positive measure,
- $\partial D_j \cap \partial\Omega = \emptyset$ $(j = 1, 2)$ or, e.g.,
- $\partial D_1 \cap \partial\Omega$ is a set of positive measure and $\partial D_2 \cap \partial\Omega = \emptyset$.

Given the vast bibliography concerning problem (1) in the case when a, b, θ are constant functions, in order to set our work in perspective we will only mention those works closely related to ours which consider the roots of f to be non-constant functions, i.e., spatially heterogeneous.

In [8] the author showed existence of patterns for the corresponding one-dimensional problem $u_t - \varepsilon^2 u_{xx} = u[\alpha(x) - u^2]$ in $(0, 1)$ under the boundary conditions $u_x(0) = u_x(1) = 0$ where $\alpha \in C(0, 1)$ is a positive function assuming a minimum on an interval $I \subset (0, 1)$ satisfying $\alpha'(x) = 0$, $x \in I$. Hence α assumes a positive minimum value all over I. The method utilized was a Brezis–Nirenberg sub-supersolution type adapted to Neumann boundary condition.

In [9] the author also studied the same unidimensional problem when α is smooth and, as opposed to [8], nondegenerate, i.e. $\alpha'' > 0$ at each local minimum of α.

In [7] the authors considered $\epsilon^2 \Delta u + (\alpha(x)^2 - u^2)u = 0$, $x \in \Omega$ with zero Neumann boundary condition in the special case $\alpha(x) = \chi_D(x)$ where $D \subset \Omega$ is a sub-domain satisfying $D = D_1 \cup D_2, \overline{D_1} \cap \overline{D_2} = \emptyset$, $\overline{\partial D \cap \Omega} \subset \Omega$ and ∂D_1, ∂D_2 are of class C^2. For sufficiently small $\epsilon > 0$, they proved existence of a local minimizer u_ϵ of the corresponding energy functional $J(u)$ on $H^1(\Omega)$ with the following asymptotic behavior: u_ϵ converges to 1 uniformly on any compact subset of D_1, converges to -1 uniformly on any compact subset of D_2 and converges to 0 uniformly on any compact subset of $\Omega \setminus (\overline{D})$.

The results we present herein generalizes [7] in many ways; in particular since we do not require $\overline{\partial D \cap \Omega} \subset \Omega$, this answer a question raised at the end of [7] as to whether this condition is necessary.

Given a function $u \in L^1(\Omega)$ we will use the following notation

$$u^{D_l} \overset{\text{def}}{=} u|_{D_l}\,, \quad l \in \{1, 2, 3\}\,.$$

Recalling that $a = \theta = b$ on D_3, our main result states as follows.

Theorem 1. *If f satisfies (f_1) and (f_2) then $\exists\, \epsilon_0 > 0$ and four families of classical stable stationary solutions $\{u_\epsilon^j\}_{0<\epsilon\le\epsilon_0}$ $(j = 1, \ldots, 4)$ to (1) such that*

- $\|u_\epsilon^1 - u_0^1\|_{L^1(\Omega)} \overset{\epsilon\to 0}{\longrightarrow} 0$ *where* $u_0^1 = a\chi_{D_1} + b\chi_{D_2} + \theta\chi_{D_3}$,
- $\|u_\epsilon^2 - u_0^2\|_{L^1(\Omega)} \overset{\epsilon\to 0}{\longrightarrow} 0$ *where* $u_0^2 = b\chi_{D_1} + a\chi_{D_2} + \theta\chi_{D_3}$,
- $\|u_\epsilon^3 - u_0^3\|_{L^1(\Omega)} \overset{\epsilon\to 0}{\longrightarrow} 0$ *where* $u_0^3 \equiv a$,
- $\|u_\epsilon^4 - u_0^4\|_{L^1(\Omega)} \overset{\epsilon\to 0}{\longrightarrow} 0$ *where* $u_0^4 \equiv b$.