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Value ranges of univalent self-mappings of the unit disc

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A B S T R A C T

We describe the value set $\{f(z_0) \,|\, f : \mathbb{D} \to \mathbb{D} \text{ univalent}, f(0) = 0, f'(0) = e^{-T}\}$, where $\mathbb{D}$ denotes the unit disc and $z_0 \in \mathbb{D} \setminus \{0\}$, $T > 0$, by applying Pontryagin's maximum principle to the radial Loewner equation.

1. Introduction and main result

Given a bounded univalent function f on a simply connected domain $\Omega \subsetneq \mathbb{C}$ and two distinct points $a, b \in \Omega$, it is quite natural to ask the question as to which values $f(b)$ can take if $f(a)$ and $f'(a)$ are prescribed. Since the Riemann mapping theorem tells us that any such domain Ω can be mapped conformally onto the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that a is mapped to 0, the problem can be restricted to the case of $\Omega = \mathbb{D}$ and $a = 0$.

By multiplying with a real constant ≤ 1 and applying an automorphism of $\mathbb{D}$, we may assume $f : \mathbb{D} \to \mathbb{D}$ and $f(0) = 0$. Then the Schwarz lemma tells us that $|f'(0)| \leq 1$ and $|f'(0)| = 1$ if and only if f is the rotation $f(z) = f'(0)z$. In order to describe the non-trivial case $|f'(0)| < 1$, we can restrict ourselves to the case $f'(0) \in (0, 1)$ because of rotational symmetry. Thus we consider the set

$$\mathcal{S}_T := \{f : \mathbb{D} \to \mathbb{D} \text{ univalent}, f(0) = 0, f'(0) = e^{-T}\}, \quad T > 0.$$

In this note, we will determine the value set

$$V_T(z_0) = \{f(z_0) : f \in \mathcal{S}_T\}, \quad z_0 \in \mathbb{D} \setminus \{0\}.$$

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Variations of the set $V_T(z_0) = \{f(z_0) : f \in \mathcal{S}_T\}$ have been determined by various authors, from the classical setting of the Schwarz and Rogosinski's lemma [8], which concerns itself with holomorphic functions $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$ that fulfill no further conditions, to a recent paper by Roth and Schleißinger [10] that determines the set $\mathcal{V}(z_0) = \{f(z_0) : f \in \mathcal{S}\}$, with the class $\mathcal{S} := \{f : \mathbb{D} \to \mathbb{D}$ univalent, $f(0) = 0, f'(0) > 0\}$. Note that $\mathcal{V}(z_0) = \cup_{T>0} V_T(z_0)$.

Our results are analogous to the results of Prokhorov and Samsonova [7], who study univalent self-mappings of the upper half-plane having the so-called hydrodynamical normalization at the boundary point ∞. Finally we note that in [2], the authors consider the set $\{\log(f(z_0)/z_0) : f : \mathbb{D} \to \mathbb{C}$ univalent, $f(0) = 0,\ |f(z)| \leq M\}$ for $M > 0$. We use a different and more straightforward approach to directly determine the set $V_T(z_0)$ by applying Pontryagin's maximum principle to the radial Loewner equation.

In the following, for the sake of simplicity, we assume that $z_0 \in (0, 1)$; for other values of z_0, we just consider the function $z \mapsto e^{i \arg z_0} f\left(e^{-i \arg z_0} z\right)$ instead of f.

Theorem 1. *Let $z_0 \in (0, 1)$. For $x_0 \in [-1, 1]$ and $T > 0$, let $r = r(T, x_0)$ be the (unique) solution to the equation*

$$\begin{aligned}(1+x_0)(1-z_0)^2\log(1-r) + (1-x_0)(1+z_0)^2\log(1+r) - (1-2x_0z_0+z_0^2)\log r \\ = (1+x_0)(1-z_0)^2\log(1-z_0) + (1-x_0)(1+z_0)^2\log(1+z_0) - (1-2x_0z_0+z_0^2)\log e^{-T}z_0\end{aligned}$$

and let

$$\sigma(T, x_0) = \frac{2(1-z_0^2)\sqrt{1-x_0^2}}{1-2x_0z_0+z_0^2}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T, x_0)\right).$$

Furthermore, for fixed $T \geq 0$, define the two curves $C_+(z_0)$ and $C_-(z_0)$ by

$$C_\pm(z_0) := \left\{w_\pm(x_0) := r(T, x_0)e^{\pm i\sigma(T,x_0)} : x_0 \in [-1, 1]\right\}.$$

Then, if $\operatorname{arctanh} z_0 < \frac{\pi}{2}$, *$V_T(z_0)$ is the closed region whose boundary consists of the two curves $C_+(z_0)$ and $C_-(z_0)$, which only intersect at $x_0 \in \{-1, 1\}$.*

For $\operatorname{arctanh} z_0 \geq \frac{\pi}{2}$, *there are two different cases: First assume that T is large enough that the equation*

$$\frac{2(1-z_0^2)\sqrt{1-x^2}}{1+2xz_0+z_0^2}\left(\operatorname{arctanh} z_0 - \operatorname{arctanh} r(T, x)\right) = \pi \tag{1.1}$$

admits a solution $x \in [-1, 1]$. Then the curves $C_+(z_0)$ and $C_-(z_0)$ intersect more than twice. There is a $\chi \in (-1, 1)$ such that $\widetilde{C}_+(z_0) \cup \widetilde{C}_-(z_0)$ is a closed Jordan curve, where

$$\widetilde{C}_\pm(z_0) := \{w_\pm(x_0) : x_0 \in [\chi, 1]\},$$

and an $\aleph \in (-1, 1)$ such that $\widehat{C}_+(z_0) \cup \widehat{C}_-(z_0)$ is a closed Jordan curve, where

$$\widehat{C}_\pm(z_0) := \{w_\pm(x_0) : x_0 \in [-1, \aleph]\}.$$

Then $V_T(z_0)$ is the closed region whose boundary is $\widetilde{C}_+(z_0) \cup \widetilde{C}_-(z_0) \cup \widehat{C}_+(z_0) \cup \widehat{C}_-(z_0)$. For smaller T that do not admit a solution to (1.1), the set $V_T(z_0)$ can be described exactly as in the case of $\operatorname{arctanh} z_0 < \frac{\pi}{2}$.

Figs. 1 and 2 show the evolution of the sets $V_T(z_0)$ over time. Note that $\operatorname{arctanh} z_0 = \frac{\pi}{2} \iff z_0 = \tanh(\pi/2) \approx 0.917$.

We prove Theorem 1 in Section 2, and in Section 3 we consider the similar problem of describing the value set $\{f^{-1}(z_0) : f \in \mathcal{S}_T$ with $z_0 \in f(\mathbb{D})\}$ for the inverse functions.