J. Math. Anal. Appl. 433 (2016) 1790–1820

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic analysis via calculus of hypergeometric functions

Petr Blaschke [1]

Mathematical Institute, Silesian University in Opava, Na Rybnicku 1, 746 01 Opava, Czech Republic

ARTICLE INFO

Article history:
Received 13 May 2015
Available online 4 September 2015
Submitted by K. Driver

Keywords:
Asymptotic analysis
Special functions
Hypergeometric functions

ABSTRACT

The generalized hypergeometric function satisfies many identities or “transforms” which can be used to establish their asymptotic behavior for large argument and even, in some cases, for large parameters. We will show that using just three transforms alone, valid for a large class of multivariate hypergeometric functions, we can use a similar “calculus” to compute asymptotic expansions even in higher dimensions.

1. Introduction

Generally speaking, there is one big drawback when dealing with asymptotic expansions (due to their extreme practical usefulness) and that is that they cannot be in general used to compute other asymptotic expansions. Once an asymptotic expansion of a function is established we throw away so many information about it (the so called exponentially small terms) that the series cannot be differentiated or integrated with respect to a parameter. In other words, there is no calculus of asymptotic expansions and every expansion has to be computed essentially *ad hoc* (methods like Laplace or saddle point are, of course, quite powerful, but they work only on integrals in very specific form with no hint how to bring an integral into this form).

The hypergeometric functions ${}_pF_q$ make an exception since their asymptotic behavior for large argument and, in many cases, even for large parameters, can be computed using some of their many “transforms” they obey. There is, for example, the Pfaff transform

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix};x\right) = (1-x)^{-b}\,{}_2F_1\left(\begin{matrix} c-a & b \\ & c \end{matrix};\frac{x}{x-1}\right), \tag{1.1}$$

the Euler transform

E-mail address: Petr.Blaschke@math.slu.cz.

[1] Supported by GA ČR grant no. 201/12/G028.

http://dx.doi.org/10.1016/j.jmaa.2015.08.066

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix};x\right) = (1-x)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a & c-b \\ & c \end{matrix};x\right), \tag{1.2} $$

and the "analytic continuation" transform: For $p = q + 1$, and $|\arg(-x)| < \pi$, assuming that $a_j \notin \mathbb{Z}\ \forall j$ and $a_j - a_k \notin \mathbb{Z}\ \forall j, k,\ j \neq k$, it holds

$$ {}_{q+1}F_q\left(\begin{matrix} a_1 & \dots & a_{q+1} \\ c_1 & \dots & c_q \end{matrix};x\right) = \sum_{i=1}^{q+1}\prod_{j\neq i}\frac{\Gamma(a_j-a_i)}{\Gamma(a_j)}\prod_{j=1}^{q}\frac{\Gamma(c_j)}{\Gamma(c_j-a_i)}(-x)^{-a_i} $$
$$ {}_{q+1}F_q\left(\begin{matrix} a_i & 1-c_1+a_i & \dots & 1-c_q+a_i \\ 1-a_1+a_i & \dots & 1-a_{q+1}-a_i \end{matrix};\frac{1}{x}\right). \tag{1.3} $$

With those in hand one can deal with great many problems concerning the Gauss hypergeometric function ${}_2F_1$. The transform (1.3) allows one to obtain asymptotic expansion for large arguments [9]. The asymptotic expansion for large parameters can be worked out using (1.1), (1.2) in many cases. The simplest one is when more of the lower parameters are large than the upper ones. In that case the asymptotic expansion is simply the Taylor series. More precisely, for $r < s$, $x \notin [1, \infty)$:

$$ {}_pF_q\left(\begin{matrix} a_1+\alpha\dots a_r+\alpha & a_{r+1}\dots a_p \\ c_1+\alpha\dots c_s+\alpha & c_{s+1}\dots c_q \end{matrix};x\right) \approx \sum_{k=0}^{\infty}\frac{(a_1+\alpha)_k\dots(a_r+\alpha)_k(a_{r+1})_k\dots(a_p)_k}{(c_1+\alpha)_k\dots(c_s+\alpha)_k(c_{s+1})_k\dots(c_q)_k}\frac{x^k}{k!}, \quad (\alpha\to+\infty), \tag{1.4} $$

where $(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}$ is the Pochhammer symbol (see [9, §7.3]). Much less is known when some parameter is large and negative even in the case of a lower parameter. Notice that for them the problem is even somewhat ill-posed, because the lower parameter c_i cannot be a negative integer (otherwise division by zero would occur). However, the Taylor series expansion looks still as an asymptotic series (consecutive terms are more and more negligible) and, indeed, in Luke's book [9, §7.3] we may find that

$$ {}_pF_{p-1}\left(\begin{matrix} a_1\dots a_p \\ c-\alpha \quad b_1\dots b_{p-2} \end{matrix};x\right) = \sum_{k=0}^{N-1}\frac{(a_1)_k\cdots(a_p)_k}{(c-\alpha)_k(b_1)_k\cdots(b_{p-2})_k}\frac{x^k}{k!} + O\left(\alpha^{-N}\right), \qquad (\alpha\to\infty), \tag{1.5} $$

valid for $\mathrm{Re}(x) < \frac{1}{2}$. And even

$$ {}_pF_{p-1}\left(\begin{matrix} a_1\dots a_p \\ c-\alpha \quad d+\alpha \quad b_1\dots b_{p-3} \end{matrix};x\right) $$
$$ = \sum_{k=0}^{N-1}\frac{(a_1)_k\cdots(a_p)_k}{(c-\alpha)_k(d+\alpha)_k(b_1)_k\cdots(b_{p-3})_k}\frac{x^k}{k!} + O\left(\alpha^{-2N}\right), \qquad (\alpha\to\infty), \tag{1.6} $$

valid for $|\arg(1-x)| < \pi$.

Overall impression is that regardless of the sign, when more large parameters are down than up, the resulting Taylor series is *always* an asymptotic expansion for some values of the argument. To this observation we refer as "More down conjecture".

Some other cases can be worked out by the aid of (1.1), (1.2). For example, the Pfaff transform (1.1) effectively solves the asymptotic expansion of the type[2]

[2] By $\approx$ we always mean an asymptotic expansion, while the symbol $\sim$ refers to a "principal term" — $f(z) \sim g(z) \Leftrightarrow \lim f(z)/g(z) = 1$.