J. Math. Anal. Appl. 433 (2016) 1821–1845

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multiple solutions to a Robin problem with indefinite weight and asymmetric reaction

Giuseppina D'Aguì [a], Salvatore A. Marano [b,*], Nikolaos S. Papageorgiou [c]

[a] *DICIEAMA, University of Messina, 98166 Messina, Italy*
[b] *Department of Mathematics and Computer Sciences, University of Catania, Viale A. Doria 6, 95125 Catania, Italy*
[c] *Department of Mathematics, National Technical University of Athens, Zografou Campus, Athens 15780, Greece*

A R T I C L E I N F O

Article history:
Received 5 June 2015
Available online 31 August 2015
Submitted by A. Cianchi

Keywords:
Robin problem
Indefinite unbounded potential
Resonance
Asymmetric crossing nonlinearity
Multiple solutions

A B S T R A C T

The existence of two nontrivial smooth solutions to a semilinear Robin problem with indefinite unbounded potential and asymmetric nonlinearity f is established. Both crossing and resonance are allowed. A third nonzero solution exists provided f is C^1. Proofs exploit variational methods, truncation techniques, and Morse theory.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ having a C^2-boundary $\partial\Omega$, let $a \in L^s(\Omega)$ for appropriate $s \geq 1$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. The semilinear elliptic equation with indefinite unbounded potential

$$-\Delta u + a(x)u = f(x, u) \quad \text{in} \quad \Omega$$

has by now been widely investigated under Dirichlet or Neumann boundary conditions; see [10,19] and [20,22], respectively, besides the references given there. If $a(x) \equiv 0$ then the case of asymmetric nonlinearities f, meaning that $t \mapsto f(x, t)t^{-1}$ crosses at least the principal eigenvalue of the relevant differential operator as t goes from $-\infty$ to $+\infty$, was also studied; cf. [6,7,24]. From a technical point of view, the Fučik spectrum is often exploited [2], which entails that the limits $\lim\limits_{t\to\pm\infty} f(x, t)t^{-1}$ do exist.

* Corresponding author.
E-mail addresses: dagui@unime.it (G. D'Aguì), marano@dmi.unict.it (S.A. Marano), npapg@math.ntua.gr (N.S. Papageorgiou).

http://dx.doi.org/10.1016/j.jmaa.2015.08.065
0022-247X/

This work treats equations having both difficulties under Robin boundary conditions. Hence, for $a(x)$ bounded only from above, $s > N$, and $\beta \in W^{1,\infty}(\partial\Omega)$ nonnegative, we consider the problem

$$\begin{cases} -\Delta u + a(x)u = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + \beta(x)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\frac{\partial u}{\partial n} := \nabla u \cdot n$, with $n(x)$ being the outward unit normal vector to $\partial\Omega$ at its point x. As usual, $u \in H^1(\Omega)$ is called a (weak) solution of (1.1) provided

$$\int_\Omega \nabla u \cdot \nabla v\, dx + \int_{\partial\Omega} \beta uv\, d\sigma + \int_\Omega auv\, dx = \int_\Omega f(x,u)v\, dx \quad \forall v \in H^1(\Omega).$$

Our assumptions on the reaction f at infinity are essentially the following.

- There exists $k \geq 2$ such that $\hat{\lambda}_k \leq \liminf_{t\to-\infty} \dfrac{f(x,t)}{t} \leq \limsup_{t\to-\infty} \dfrac{f(x,t)}{t} \leq \hat{\lambda}_{k+1}$,
- $\limsup_{t\to+\infty} \dfrac{f(x,t)}{t} \leq \hat{\lambda}_1$, and $\lim_{t\to+\infty} \left[f(x,t)t - 2\int_0^t f(x,\tau)d\tau \right] = +\infty$

uniformly in $x \in \Omega$. Here, $\hat{\lambda}_n$ denotes the nth-eigenvalue of the problem

$$-\Delta u + a(x)u = \lambda u \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n} + \beta(x)u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

It should be noted that a possible interaction (resonance) with eigenvalues is allowed. If an additional condition on the behavior of $t \mapsto f(x,t)t^{-1}$ as $t \to 0$ holds then we obtain at least two nontrivial C^1-solutions to (1.1), one of which is positive; see Theorems 3.1–3.3 for precise statements. As an example, Theorem 3.1 applies when

$$f(x,t) := \begin{cases} bt & \text{if } t \leq 1, \\ \hat{\lambda}_1 t - \sqrt{t} + (b - \hat{\lambda}_1 + 1)t^{-1} & \text{otherwise}, \end{cases}$$

with $\hat{\lambda}_k \leq b \leq \hat{\lambda}_{k+1}$ and $k > 2$ large enough, or

$$f(x,t) := \begin{cases} b(t+1) - c & \text{if } t < -1, \\ ct & \text{if } |t| \leq 1, \\ \hat{\lambda}_1(t-1) + c & \text{otherwise}, \end{cases}$$

where $c > \hat{\lambda}_2$. Let us point out that, unlike previous results, the nonlinearities treated by Theorem 3.3 turn out to be concave near zero. Finally, Theorem 3.4 gives a third nontrivial C^1-solution once

$$f(x,\cdot) \in C^1(\mathbb{R}) \quad \text{and} \quad \sup_{t\in\mathbb{R}} |f'_t(\cdot,t)| \in L^\infty(\Omega).$$

Our arguments are patterned after those of [13] (cf. also [12]) where, however, the Dirichlet problem is investigated, $a(x) \equiv 0$, but the p-Laplace operator $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ appears. Moreover, the hypotheses on f made there do not permit resonance at any eigenvalue. The approach we adopt exploits variational and truncation techniques, as well as results from Morse theory. Regularity of solutions basically arises from [26].