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Harmonic functions which vanish on a cylindrical surface

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A R T I C L E I N F O

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A B S T R A C T

Suppose that a harmonic function h on a finite cylinder vanishes on the curved part of the boundary. This paper answers a question of Khavinson by showing that h then has a harmonic continuation to the infinite strip bounded by the hyperplanes containing the flat parts of the boundary. The existence of this extension is established by an analysis of the convergence properties of a double series expansion of the Green function of an infinite cylinder beyond the domain itself.

1. Introduction

The Schwarz reflection principle gives a formula for extending a harmonic function h on a domain $\Omega \subset \mathbb{R}^N$ through a relatively open subset E of the boundary $\partial\Omega$ on which h vanishes, provided E lies in a hyperplane (and is a relatively open subset thereof). By the Kelvin transformation there is a corresponding result where E lies in a sphere. When $N = 2$, such a reflection principle holds also when E is contained in an analytic arc (see Chapter 9 of [7]). However, when $N \geq 3$ and N is odd, Ebenfelt and Khavinson [4] (see also [6] and Chapter 10 of [7]) have shown that a reflection law can only hold when the containing real analytic surface is either a hyperplane or a sphere.

Now let $N \geq 3$, let Ω_a be the finite cylinder $B' \times (-a, a)$, where B' is the open unit ball in $\mathbb{R}^{N-1}$ and $a > 0$, and let $\Omega = B' \times \mathbb{R}$. Dima Khavinson raised the following question with the authors:

Question. Given a harmonic function h on Ω which vanishes on $\partial\Omega$, does it follow that h must have a harmonic extension to $\mathbb{R}^N$?

Although the above results show that there can be no pointwise reflection formula for such an extension, this paper will establish that such an extension does indeed exist.

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We will use the notation $x = (x', x_N)$ to denote a typical point of $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$.

Theorem 1. *Let h be a harmonic function on Ω_a which continuously vanishes on $\partial B' \times (-a, a)$. Then h has a harmonic extension $\widetilde{h}$ to $\mathbb{R}^{N-1} \times (-a, a)$. Further, for any $b \in (0, a)$, there is a constant c, depending on a, b, N and h, such that*

$$\left|\widetilde{h}(x)\right| \le c \left\|x'\right\|^{1-N/2} \qquad (x' \in \mathbb{R}^{N-1} \backslash B', |x_N| < b). \tag{1}$$

It is a classical fact that the Green function for a three-dimensional infinite cylinder can be represented as a double series involving Bessel functions and Chebychev polynomials: see, for example, p. 62 of Dougall [3] or p. 78 of Carslaw [2]. Our approach to proving Theorem 1 involves establishing such a double series representation in N dimensions and analysing its convergence properties outside the cylinder.

2. Preparatory material

Let J_ν and Y_ν denote the usual Bessel functions of order $\nu \ge 0$ of the first and second kinds (see Watson [12]). Thus these functions both satisfy the differential equation

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) y = 0. \tag{2}$$

Further, let $(j_{\nu,m})_{m \ge 1}$ denote the sequence of positive zeros of J_ν, in increasing order. We collect below some facts that we will need.

Lemma 2. *(i)* $\dfrac{d}{dz} z^\nu J_\nu(z) = z^\nu J_{\nu-1}(z)$ *and* $\dfrac{d}{dz} \dfrac{J_\nu(z)}{z^\nu} = -\dfrac{J_{\nu+1}(z)}{z^\nu}$.

(ii) $J_{\nu-1}(z) + J_{\nu+1}(z) = \dfrac{2\nu J_\nu(z)}{z}$ *and* $J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z)$.

(iii) $J_\nu(t) Y'_\nu(t) - Y_\nu(t) J'_\nu(t) = \dfrac{2}{\pi t}$ $(t > 0)$.

(iv) $\{J_\nu(t)\}^2 + \{Y_\nu(t)\}^2 < \dfrac{2}{\pi} \left(t^2 - \nu^2\right)^{-1/2}$ $(t > \nu \ge \frac{1}{2})$.

(v) $|J_\nu(t)| \le \left(\dfrac{t}{2}\right)^\nu \dfrac{1}{\Gamma(\nu+1)}$ $(t \ge 0)$.

(vi) $j_{0,m} \ge (m + 3/4)\pi$.

(vii) $j_{\nu,m} \ge j_{0,m} + \nu$.

(viii) $|J_\nu(t)| < \nu^{-1/3}$ $(\nu > 0, t \ge 0)$.

(ix) $|J_\nu(t)| \le \min\{1, t^{-1/3}\}$ $(t > 0)$.

(x) $\{J_\nu(t)\}^2 + \{Y_\nu(t)\}^2 < \dfrac{2}{\pi t}$ $(0 \le \nu \le \frac{1}{2}, t > 0)$.

Proof. (i) and (ii). See p. 45 of Watson [12].

(iii) See p. 76, (1) of [12].

(iv) See p. 447, (1) of [12].

(v) See p. 49, (1) of [12].

(vi) See p. 489 of [12].

(vii) See Laforgia and Muldoon [8], (2.4).

(viii) See Landau [9].

(ix) We know from p. 406, (10) of [12] that $|J_\nu| \le 1$, and from [9] that $|J_\nu(t)| \le t^{-1/3}$.

(x) By Section 13.74 of [12] the function $t \mapsto t\left(\{J_\nu(t)\}^2 + \{Y_\nu(t)\}^2\right)$ is non-decreasing when $0 \le \nu \le \frac{1}{2}$, and has limit $2/\pi$ at ∞. □