J. Math. Anal. Appl. 433 (2016) 1883–1903

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Effective asymptotic regularity for one-parameter nonexpansive semigroups

Ulrich Kohlenbach *, Angeliki Koutsoukou-Argyraki

Department of Mathematics, Technische Universität Darmstadt, Schlossgartenstraße 7, 64289 Darmstadt, Germany

ARTICLE INFO

Article history:
Received 15 July 2015
Available online 28 August 2015
Submitted by T. Domínguez Benavides

Keywords:
Nonexpansive semigroup
Rate of asymptotic regularity
Common fixed point
Krasnoselskii iteration
Proof mining

ABSTRACT

We give explicit bounds on the computation of approximate common fixed points of one-parameter strongly continuous semigroups of nonexpansive mappings on a subset C of a general Banach space. Moreover, we provide the first explicit and highly uniform rate of convergence for an iterative procedure to compute such points for convex C. Our results are obtained by a logical analysis of the proof (proof mining) of a theorem by T. Suzuki.

1. Introduction and preliminaries

In this paper we give a quantitative analysis of a theorem due to Suzuki [18] which states that in order to compute a common fixed point of a one-parameter strongly continuous semigroup of nonexpansive mappings it is sufficient to compute a fixed point of a single nonexpansive mapping which is derived from this semigroup. As a corollary to this we get an explicit and highly uniform rate of asymptotic regularity for the semigroup. Such semigroups play a central role in the study of abstract Cauchy problems (see e.g. [5,3,16,2] for the classical theory).

Suzuki's proof which we analyze here is not effective and is based on a number-theoretic density result. This makes our extraction of explicit bounds highly nontrivial and so our paper is also a significant new contribution to the so-called 'Proof Mining' program (going back to pioneering ideas of Georg Kreisel in the 50s) which uses tools from logic (applied proof theory) to extract new quantitative constructive information by logical analysis of prima facie noneffective proofs. The information is 'hidden' behind an implicit use of

* Corresponding author.
E-mail addresses: kohlenbach@mathematik.tu-darmstadt.de (U. Kohlenbach), koutsoukou@mathematik.tu-darmstadt.de (A. Koutsoukou-Argyraki).

http://dx.doi.org/10.1016/j.jmaa.2015.08.059
0022-247X/

quantifiers in the proof, and its extraction is guaranteed by certain logical metatheorems if the statement proved is of a certain logical form (for instance here a $\forall\exists$ statement) and proved within a suitable deductive framework [8,6,9]. The resulting quantitative form of the given theorem then comes again with an ordinary proof in analysis (as in this paper) which makes no reference to any tools from logic. Within the past 15 years, proof mining has been applied by the first author and his collaborators to various fields of mathematics, including approximation theory, ergodic theory, fixed point theory, nonlinear analysis, and (recently) PDE theory (see e.g. [10,9,11,13]).

In this section we recall some basic definitions, introduce certain preliminary concepts and state our main result.

By $\mathbb{N}$ we denote the set of natural numbers $\{1, 2, \ldots\}$, by $\mathbb{Z}$ the set of integers and by $\mathbb{Z}^+$, $\mathbb{Q}^+$, $\mathbb{R}^+$ the sets of nonnegative integers, rationals and reals respectively.

Definition 1. Given a Banach space E and a subset $C \subseteq E$, a mapping $T : C \to E$ is nonexpansive if

$$\forall x, y \in C \ (\|Tx - Ty\| \leq \|x - y\|) \,.$$

Definition 2. A family $\{T(t) : t \geq 0\}$ of self-mappings $T(t) : C \to C$ for a subset C of a Banach space E is called a one-parameter strongly continuous semigroup of nonexpansive mappings (or *nonexpansive semigroup* for short) if the following conditions hold:

(1) for all $t \geq 0$, $T(t)$ is a nonexpansive mapping on C,
(2) $T(s) \circ T(t) = T(s + t)$,
(3) for each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is continuous.

Our main result will be a quantitative version of the following theorem by Suzuki in [18]:

Theorem 1. *(Theorem 1 in [18]) Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on a subset $C \subseteq E$ for some Banach space E. Let $F(T(t))$ denote the set of fixed points of $T(t)$. Let $\alpha, \beta \in \mathbb{R}^+$ satisfying $\alpha/\beta \in \mathbb{R}^+ \setminus \mathbb{Q}^+$. Then for all $\lambda \in (0, 1)$ we have:*

$$\bigcap_{t \geq 0} F(T(t)) = F(\lambda T(\alpha) + (1 - \lambda)T(\beta)),$$

where

$$\lambda T(\alpha) + (1 - \lambda)T(\beta)$$

is a mapping from C into E defined by

$$(\lambda T(\alpha) + (1 - \lambda)T(\beta))x = \lambda T(\alpha)x + (1 - \lambda)T(\beta)x$$

for $x \in C$.

The inclusion

$$\bigcap_{t \geq 0} F(T(t)) \subseteq F(\lambda T(\alpha) + (1 - \lambda)T(\beta))$$

is trivial.