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On the ideal compressible magnetohydrodynamic equations

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ABSTRACT

In this paper, we consider the short time strong solutions to the ideal compressible magnetohydrodynamic equations with initial vacuum, where the velocity field satisfies the Navier-slip condition. Inspired by Kato and Lax's idea, we use the contraction mapping argument to prove the local existence. Moreover, under the Navier-slip condition, we establish a criterion for possible breakdown of such solutions at finite time in terms of the temporal integral of L^∞ norm of the deformation tensor $D(u)$.

1. Introduction

For its wide range of applications in physical objects, such as astrophysics, geophysics and plasma physics, the magnetohydrodynamic (MHD) model has been studied both from a theoretical and numerical perspective. In this paper, we study the following compressible MHD flows in the Eulerian coordinates:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + \operatorname{curl} H \times H, \\ \partial_t H - \operatorname{curl}(u \times H) = 0, \quad \operatorname{div} H = 0. \end{cases} \tag{1.1}$$

The system (1.1) is called ideal MHD equations. Here ρ, u and H denote the density, velocity and magnetic field, respectively. The constant μ is the shear viscosity coefficient satisfying the physical restriction: $\mu > 0$. The pressure P is a given state equation, we assume that $P(0) = 0$ and:

$$P : [0, \infty) \to \mathbb{R} \quad \text{is a locally Lipschitz continuous function.} \tag{1.2}$$

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In this work, we investigate the ideal compressible MHD equations (1.1) in a bounded, simply connected, smooth domain Ω, with initial data

$$(\rho, u, H)\big|_{t=0} = \big(\rho_0(x), u_0(x), H_0(x)\big), \tag{1.3}$$

and the Navier-slip boundary condition

$$u \cdot n|_{\partial\Omega} = 0, \tag{1.4}$$

$$\operatorname{curl} u \times n|_{\partial\Omega} = 0, \tag{1.5}$$

where n is the unit outer normal vector to $\partial\Omega$.

Throughout this paper, we give some notations by reason of the convenience of discussions:

$$\int f dx = \int\limits_{\Omega} f dx, \qquad \int\limits_0^t \int f dx ds = \int\limits_0^t \int\limits_{\Omega} f dx ds.$$

For $1 \le r \le \infty$, we denote the L^r spaces and the standard Sobolev spaces as follows

$$L^r = L^r(\Omega), \qquad H^k = W^{k,2} = W^{k,2}(\Omega).$$

In the recent years, there are lots of literatures on the MHD system by physicists and mathematicians, due to its physical importance, complex, rich phenomena and mathematical challenges. The mathematical results are mainly concerned with the full MHD system, see Refs. [2,6–10,20,25,27,28,31] and the references therein. For example, Li et al. [20] showed the global well-posedness of classical solution, where the flow density is allowed to contain vacuum states. However, there is few results on the ideal MHD system except for [7,19,30].

The system (1.1) implies that in a highly conducting fluid the magnetic field lines move along exactly with the fluid, rather than simply diffusing out. Being dissipationless, the ideal MHD equations are conservative and this leads to some powerful theorems and simple physical properties. Further, since the ideal MHD equations are so much more amenable to mathematical analysis, they can be used to investigate realistic geometries. Notwithstanding the wide applicability of ideal MHD in space and laboratory plasma physics a lot of caution needs to be sounded over results derived from it.

An appropriate choice of boundary conditions is a big and important problem in describing the dynamics of continuum fluid. In many theoretical studies, the well-accepted hypothesis is that if the boundary is impermeable, then the fluid adheres completely to it: no-slip boundary condition (i.e. Dirichlet boundary condition). While, in 1823, Navier [22] proposed a slip boundary condition and claimed that the component of the fluid velocity tangent to the surface should be proportional to the rate of strain at the surfaces. This boundary condition is referred to as characteristic since it consists of stream lines at all times. As pointed in [24]: "However, the use of the no-slip condition is being questioned due to recent developments in microfluidic. Several experiments have ... claimed that in many cases the fluid velocity satisfies a Navier-slip condition at the boundary." To the authors' knowledge, most of the above work concern on the whole space $\mathbb{R}^3$ or Dirichlet boundary condition. There is few available results for the ideal compressible MHD equations with velocity satisfying the Navier-slip condition. Therefore, it is interesting and meaningful to consider the ideal MHD system with the Navier-slip condition.

The aim of this paper is to prove the existence of unique local strong solutions to (1.1)–(1.5) with $\inf \rho_0 = 0$, and investigate the blow-up mechanism. Before stating the main theorems, we first give the definition of strong solutions.