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New results for triangular reaction cross diffusion system

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ABSTRACT

We present an approach based on entropy and duality methods for “triangular” reaction cross diffusion systems of two equations, in which cross diffusion terms appear only in one of the equations. Thanks to this approach, we recover and extend many existing results on the classical “triangular” Shigesada–Kawasaki–Teramoto model.

1. Introduction

Reaction cross diffusion equations naturally appear in physics (cf. [4] for example) as well as in population dynamics. We are interested here in the study of a class of systems first introduced by Shigesada, Kawasaki, and Teramoto (cf. [22]). Those systems aim at modeling the repulsive effect of populations of two different species in competition, and are possibly leading to the apparition of patterns (cf. [14]).

The unknowns are the quantities $u := u(t,x) \geq 0$ and $v := v(t,x) \geq 0$. They represent the number densities of the two considered species (say, species 1 and species 2). They depend on the time variable $t \in \mathbb{R}_+$ and the space variable $x \in \Omega$. Hereafter, Ω is a smooth bounded domain of $\mathbb{R}^N$ $(N \in \mathbb{N}^* := \mathbb{N} - \{0\})$ and we denote by $n = n(x)$ its unit normal outward vector at point $x \in \partial\Omega$. The original model of [22] writes

$$\begin{cases} \partial_t u - \Delta_x(d_u\, u + d_{11}\, u^2 + d_{12}\, u\, v) = u\,(r_u - r_a\, u - r_b\, v) & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_t v - \Delta_x(d_v\, v + d_{21}\, u\, v + d_{22}\, v^2) = v\,(r_v - r_c\, v - r_d\, u) & \text{in } \mathbb{R}_+ \times \Omega, \\ \nabla_x u \cdot n = \nabla_x v \cdot n = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{1}$$

The coefficients $r_u, r_v > 0$ are the growth rates in absence of other individuals, $r_a, r_b, r_c, r_d > 0$ correspond to the logistic inter- and intraspecific competition effects, and $d_u, d_v > 0$ are the diffusion rates. The coefficients

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$d_{ij} \geq 0$ $(i, j = 1, 2)$ represent the repulsive effect: individuals of species i increase their diffusion rate in presence of individuals of their own species when $d_{ii} > 0$ (self diffusion) or of the other species when $d_{ij} > 0$ ($i \neq j$, cross diffusion).

In the sequel, we shall only consider the case when $d_{21} = 0$ and $d_{12} > 0$, which is sometimes called "triangular". In such a situation, the second equation is coupled to the first one only through the competition (reaction) term while the first one is coupled to the second one through both diffusion and competition terms (the fully coupled system when $d_{21} > 0$ and $d_{12} > 0$ has a quite different mathematical structure, cf. [6] and [11] for example). We shall also only focus on the case when no self diffusion appears (that is $d_{11} = d_{22} = 0$) since this case is the most studied one: note however that the presence of self-diffusion (that is, $d_{11} > 0$ and/or $d_{22} > 0$) usually helps to obtain better bounds on the solution. As a consequence, our results are expected to hold when self-diffusion is present.

Under the extra assumptions detailed above, the Shigesada–Kawasaki–Teramoto system writes

$$\begin{cases} \partial_t u - \Delta_x(d_u\, u + d_{12}\, u\, v) = u\,(r_u - r_a\, u - r_b\, v) & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_t v - d_v\, \Delta_x v = v\,(r_v - r_c\, v - r_d\, u) & \text{in } \mathbb{R}_+ \times \Omega, \\ \nabla_x u \cdot n = \nabla_x v \cdot n = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{2}$$

Following [13], this system can be seen as the formal singular limit of a reaction diffusion system which writes

$$\begin{cases} \partial_t u_A^\varepsilon - d_u\, \Delta_x u_A^\varepsilon = [r_u - r_a\,(u_A^\varepsilon + u_B^\varepsilon) - r_b\, v^\varepsilon]\, u_A^\varepsilon + \dfrac{1}{\varepsilon}[k(v^\varepsilon)\, u_B^\varepsilon - h(v^\varepsilon)\, u_A^\varepsilon] & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_t u_B^\varepsilon - (d_u + d_B)\, \Delta_x u_B^\varepsilon = [r_u - r_a\,(u_A^\varepsilon + u_B^\varepsilon) - r_b\, v^\varepsilon]\, u_B^\varepsilon - \dfrac{1}{\varepsilon}[k(v^\varepsilon)\, u_B^\varepsilon - h(v^\varepsilon)\, u_A^\varepsilon] & \text{in } \mathbb{R}_+ \times \Omega, \\ \partial_t v^\varepsilon - d_v\, \Delta_x v^\varepsilon = [r_v - r_c\, v^\varepsilon - r_d\,(u_A^\varepsilon + u_B^\varepsilon)]\, v^\varepsilon & \text{in } \mathbb{R}_+ \times \Omega, \\ \nabla_x u_A^\varepsilon \cdot n = \nabla_x u_B^\varepsilon \cdot n = \nabla_x v^\varepsilon \cdot n = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{3}$$

where $d_B > 0$, and h, k are two (continuous) functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying (for all $v \geq 0$) the identity

$$d_B\, \frac{h(v)}{h(v) + k(v)} = d_{12}\, v.$$

The limit holds (at the formal level) in the following sense: if u_A^ε, u_B^ε, and v^ε are solutions to system (3) (with ε-independent initial data), the quantity $(u_A^\varepsilon + u_B^\varepsilon, v^\varepsilon)$ converges towards (u, v), where u and v are solutions to system (2). Note that this asymptotics can be biologically meaningful: when $\varepsilon > 0$, the system (3) represents a microscopic model in which the species u can be found in two states (the quiet state u_A and the stressed state u_B), and the individuals of this species switch from one state to the other one with a "large" rate (proportional to $1/\varepsilon$).

We present in this paper results for the existence, uniqueness and stability of a large class of systems including (2). More precisely, we relax the assumption stating that the competition terms are logistic (quadratic), and replace it with the assumption stating that the competition terms are given by power laws (the powers being suitably chosen). We also relax the assumption stating that the cross diffusion term is quadratic (that is, proportional to $u\,v$) and replace it by the more general assumption stating that it writes $u\,\phi(v)$ (with $\phi \in C^1(\mathbb{R}_+)$, and ϕ nonnegative).

Hence, we shall consider the system

$$\partial_t u - \Delta_x(d_u\, u + u\,\phi(v)) = u\,(r_u - r_a\, u^a - r_b\, v^b) \qquad \text{in } \mathbb{R}_+ \times \Omega, \tag{4}$$

$$\partial_t v - d_v\, \Delta_x v = v\,(r_v - r_c\, v^c - r_d\, u^d) \qquad \text{in } \mathbb{R}_+ \times \Omega, \tag{5}$$