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Spectral invariants of periodic nonautonomous discrete dynamical systems

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ABSTRACT

For an interval map, the poles of the Artin–Mazur zeta function provide topological invariants which are closely connected to topological entropy. It is known that for a time-periodic nonautonomous dynamical system F with period p, the p-th power $[\zeta_F(z)]^p$ of its zeta function is meromorphic in the unit disk. Unlike in the autonomous case, where the zeta function $\zeta_f(z)$ only has poles in the unit disk, in the p-periodic nonautonomous case $[\zeta_F(z)]^p$ may have zeros. In this paper we introduce the concept of spectral invariants of p-periodic nonautonomous discrete dynamical systems and study the role played by the zeros of $[\zeta_F(z)]^p$ in this context. As we will see, these zeros play an important role in the spectral classification of these systems.

1. Motivation and statement of main results

A large part of mathematical models appearing in discrete dynamical systems for approaching applied situations coming from Biology, Physics, Economy, . . . are governed by a single map. However too often this approach is not realistic because there are a lot of processes involving different responses according to the different steps of them. Thus it is necessary to model these systems with more than a map.

Nonautonomous discrete dynamical systems generated by p-periodic sequences of maps, called in literature as p-periodic discrete systems or p-periodic difference equations, have been attracting the attention of many authors, see e.g. among others: [2,5,8–14,18].

The notion of topological entropy of a nonautonomous discrete dynamical system, introduced by Kolyada and Snoha in [16], is deeply related with the main goal of this work: the study of the periodic structure of

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a one-dimensional p-periodic discrete system from the spectral point of view. That is, we are interested in the invariants of a p-periodic system, F, that can be detected by the analytic properties of its zeta function, $\zeta_F(z)$ (an analogue of the Artin–Mazur zeta function of an autonomous system).

In [17], Milnor and Thurston proved that, under some instability conditions, the Artin–Mazur zeta function, $\zeta_f(z)$, of an autonomous system generated by a continuous and piecewise monotone interval map f is meromorphic in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Meanwhile, an analogue of this theorem for nonautonomous periodic discrete systems was presented in [4]: it was proved that, for a p-periodic discrete system F, under similar instability conditions, the function $[\zeta_F(z)]^p$ (the p-th power of the zeta function of F) is meromorphic in D.

In the same paper it was also observed that, unlike in the autonomous case where the meromorphic function $\zeta_f(z)$ has no zeros in D, in the nonautonomous p-periodic case the function $[\zeta_F(z)]^p$ may exhibit zeros. Since the poles of $[\zeta_F(z)]^p$ are invariants of the system, the so-called spectral invariants, the main objective of this paper is to understand the role played by these zeros in its spectral classification.

A brief but more detailed description of the main ideas of [4] will help us to understand the context of the main results of this paper.

Let $I \subset \mathbb{R}$ be a compact interval and $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers. A continuous map $f : I \to I$ is said to be piecewise monotone (for short a *cpm* map on I) providing that there exists a finite family of subintervals $J_1, \ldots, J_k \subseteq I$ covering I such that f is strictly monotonic on each J_i. As usual the n-th iterate, $f^{\circ n}$, of a *cpm* map $f : I \to I$ is defined as the composition of f with itself n times.

By a discrete dynamical system on I (for short a discrete system) we mean a sequence $(f_i)_{i\in\mathbb{N}}$ of *cpm* maps on I satisfying the following condition: Every crossing set

$$C_{ij} = \{x \in I : f_i(x) = f_j(x)\}, \text{ for } i, j \in \mathbb{N}, \tag{1}$$

is a finite union of compact intervals.[1]

A discrete system $(f_i)_{i\in\mathbb{N}}$ is called p-periodic, if the sequence of maps $(f_i)_{i\in\mathbb{N}}$ is periodic with minimal period $p \in \mathbb{Z}^+$. A 1-periodic discrete system is also called autonomous, otherwise we call it nonautonomous. By abuse of language we use the same symbol, f, to denote a *cpm* map $f : I \to I$ and the corresponding autonomous discrete system. We reserve the symbol F to denote any discrete system (periodic or not).

Let $F = (f_i)_{i\in\mathbb{N}}$ be a discrete system. For each $n \in \mathbb{N}$ one defines the n-th iterate of F as the *cpm* map $g_n : I \to I$ given inductively by

$$g_0 = id_I \text{ and } g_{n+1} = f_n \circ g_n \text{ for } n \in \mathbb{N}.$$

The trajectory of a point $x \in I$ under F, is the sequence $(x_i)_{i=0}^{+\infty}$, defined by $x_i = g_i(x)$ for all $i \in \mathbb{N}$. A point $x \in I$ is called a periodic point of F with period $n \in \mathbb{Z}^+$ if the trajectory of x is a periodic sequence and n is one (not necessarily the smallest one) of its periods. We write $\mathrm{Per}(F, n)$ for denoting the set of periodic points of F with period $n \in \mathbb{Z}^+$ and $\mathrm{Per}(F)$ for the set of periodic points of F, i.e.,

$$\mathrm{Per}(F, n) = \{x \in I : g_{i+n}(x) = g_i(x) \text{ for all } i \in \mathbb{N}\}$$

and

$$\mathrm{Per}(F) = \bigcup_{n\in\mathbb{Z}^+} \mathrm{Per}(F, n).$$

Notice that, the inclusion

[1] Degenerated compact intervals, $J = \emptyset$ or $J = \{c\}$, are allowed. Hence, some of the crossing sets of a p-piecewise monotone dynamical system can be finite or even empty.