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Characterizations of automorphisms of operator algebras on Banach spaces

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ABSTRACT

Let X be a complex Banach space of dimension greater than one, and denote by $B(X)$ the algebra of all the bounded linear operators on X. It is shown that if $\phi : B(X) \to B(X)$ is a multiplicative map (not assumed linear) and if ϕ is sufficiently close to a linear automorphism of $B(X)$ in some uniform sense, then it is actually an automorphism.

1. Introduction

It is a classical result [3] that every algebra automorphism ϕ of $B(X)$, the algebra of all bounded linear operators on a Banach space X, is spatial, i.e., there exists an invertible operator T in $B(X)$ such that $\phi(A) = TAT^{-1}$ for all $A \in B(X)$. There are two main directions in characterizing automorphisms of $B(X)$. One is to study when a linear map is an automorphism [2,4,8,11,12,16–18]. Among other results, Jafarian and Sourour [8] showed that a surjective linear map of $B(X)$ preserves spectrum if and only if it is either an automorphism or an anti-automorphism. Using this, Larson and Sourour [12] proved that each surjective local automorphism of $B(X)$ is actually an automorphism when X is infinite-dimensional, which implies that the automorphism space of $B(X)$ is reflexive.

Another direction of characterizing automorphisms of $B(X)$ is to investigate when a multiplicative map (not assumed linear) is an automorphism [1,7,13–15]. Semrl [15] considered a bijection φ of $B(X)$ satisfying $\|\varphi(AB) - \varphi(A)\varphi(B)\| < \epsilon$ for all $A, B \in B(X)$, and showed that φ is either an automorphism or an anti-automorphism in the case X is infinite-dimensional. Let H be a Hilbert space. Molnar [14] showed that a continuous multiplicative map of $B(H)$ which preserves co-rank is a linear automorphism or a

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conjugate-linear automorphism. Recently, Marcoux, Radjavi and Sourour [13] proved that a multiplicative map $\varphi : B(H) \to B(H)$ (not assumed linear and bijective) which is sufficiently close to a linear automorphism ψ, i.e. there exists a $0 < \delta < \frac{1}{4}$ such that $\|\varphi(A) - \psi(A)\| < \delta\|\psi(A)\|$ for all $A \in B(H)$, then φ is an automorphism.

The aim of the present paper is to establish the result of [13] mentioned above in the Banach space setting. Moreover, the upper bound of δ is expanded, namely, the map can be farther away from the (fixed) automorphism. Because of the obvious difference between the Hilbert space and the Banach space, our approach is very different from that in [13]. In fact, our result is closely linked with projection constants.

For a closed subspace E of X, we let

$$\lambda(X, E) = \inf\{\|P\| : P \in B(X) \text{ is a projection with range } E\},$$

and $\lambda(X, E) = \infty$ if there are no projections with range E. Further, for $n \in \mathbb{N}$, let

$$\lambda_n(X) = \sup\{\lambda(X, E) : E \text{ is an } n\text{-dimensional subspace of } X\},$$

and call it the n-dimensional projection constant of X. Obviously, $\lambda_n(H) = 1$ for any Hilbert space H. Due to Kadec and Snobar [9], $\lambda_n(X) \le \sqrt{n}$ for any Banach space X. For more information, see, for example, [5,6,10].

Let's now introduce some terminologies. Throughout, X is a complex Banach space with topological dual X^*. Denote by $B(X)$ the algebra of all bounded linear operators on X. For $A \in B(X)$, by A^* denote the adjoint of A, by $\operatorname{im} A$ the range of A, by $\operatorname{rank} A$ the dimension of $\operatorname{im} A$. For $x \in X,\ f \in X^*$, the rank at most one operator $x \otimes f$ is defined by $x \otimes f(z) = f(z)x$. A projection P is an operator in $B(X)$ satisfying $P^2 = P$.

The following lemma can be found in any textbook of functional analysis and we omit the proof here.

Lemma 1.1 *(Riesz lemma). For a non-dense subspace E of X, given $0 < r < 1$, there is $x \in X$ with $\|x\| = 1$ but* $\operatorname{dist}(x,\ E) = \inf_{y\in E}\|x - y\| > r$.

We close this section with a result about the rank of projections. This is surely known, but we include a proof for completeness.

Lemma 1.2. *Suppose P and Q are projections in $B(X)$. If $\|P - Q\| < 1$ then* $\operatorname{rank} P = \operatorname{rank} Q$.

Proof. Let $M = \operatorname{im} P = \{Px : x \in X\}$. For $y \in M$, we have $Py = y$ and hence $\|Qy\| \ge \|Py\| - \|(P-Q)y\| \ge (1 - \|P - Q\|)\|y\|$. This shows that the restriction $Q|_M$ of Q to M is injective, which further implies that $\operatorname{rank} Q \ge \dim M = \operatorname{rank} P$. Similarly, $\operatorname{rank} P \ge \operatorname{rank} Q$. The desired equality follows. □

2. Auxiliary lemmas

Throughout this section, X is a complex Banach space of dimension greater than one. By $\mathcal{P}_0(X)$ and $\mathcal{P}_1(X)$, we denote, respectively, the set of all projections of rank one in $B(X)$ and the set of all projections of rank one with norm 1 in $B(X)$. We always suppose that ϕ is a multiplicative map from $B(X)$ into itself and satisfies $\|\phi(A) - A\| \le \delta\|A\|$ for all $0 \ne A \in B(X)$, where $\delta \in (0, 1)$ is fixed.

Lemma 2.1. *ϕ preserves rank-1 operators.*

Proof. For non-zero vectors $x \in X$ and $f \in X^*$, choose $g \in X^*$ such that $g(x) = 1$ and $\|g\| = \frac{1}{\|x\|}$. Let $P = x \otimes g$. Then $P \in \mathcal{P}_1(X)$. Therefore, $\phi(P)$ is a projection and $\|\phi(P) - P\| < \|P\| = 1$. It follows from