J. Math. Anal. Appl. 430 (2015) 262–278

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# A generalisation of the fractional Brownian field based on non-Euclidean norms

Ilya Molchanov [a,*], Kostiantyn Ralchenko [b]

[a] University of Bern, Institute of Mathematical Statistics and Actuarial Science, Sidlerstrasse 5, CH-3012 Bern, Switzerland

[b] Taras Shevchenko National University of Kyiv, Department of Probability Theory, Statistics and Actuarial Mathematics, Volodymyrska 64/13, 01601 Kyiv, Ukraine

ARTICLE INFO

*Article history:*
Received 10 December 2014
Available online 30 April 2015
Submitted by U. Stadtmueller

*Keywords:*
Fractional Brownian field
Fractional Poisson field
Radial $p$th mean body
Polar projection body
Minkowski space
Star body

ABSTRACT

We explore a generalisation of the Lévy fractional Brownian field on the Euclidean space based on replacing the Euclidean norm with another norm. A characterisation result for admissible norms yields a complete description of all self-similar Gaussian random fields with stationary increments. Several integral representations of the introduced random fields are derived. In a similar vein, several non-Euclidean variants of the fractional Poisson field are introduced and it is shown that they share the covariance structure with the fractional Brownian field and converge to it. The shape parameters of the Poisson and Brownian variants are related by convex geometry transforms, namely the radial $p$th mean body and the polar projection transforms.



## 1. Introduction

The *multiparameter fractional Brownian motion* or the *Lévy fractional Brownian field* (fBf) with Hurst index $H \in (0,1)$ is a centred Gaussian random field $X(z)$, $z \in \mathbb{R}^d$, with the covariance function

$$\mathbf{E}[X(z_1)X(z_2)] = \frac{1}{2}\left[\|z_1\|^{2H} + \|z_2\|^{2H} - \|z_1 - z_2\|^{2H}\right], \tag{1}$$

where $\|z\|$ is the Euclidean norm of $z$. If $h = \frac{1}{2}$, this yields the Lévy Brownian motion on $\mathbb{R}^d$. If $d = 1$, one recovers the classical univariate fractional Brownian motion (fBm), see [25]. This random field was introduced by A.M. Yaglom [35] as a model of turbulence in fluid mechanics. Various proofs showing that (1) defines a valid covariance function are given in [11,25,29,30]. Further results including series expansions and a general functional limit theorem can be found in [24]. The two most important integral representations of the Lévy fBf are the moving average representation using the integral with respect to the white noise

* Corresponding author.
*E-mail addresses:* ilya.molchanov@stat.unibe.ch (I. Molchanov), k.ralchenko@gmail.com (K. Ralchenko).

http://dx.doi.org/10.1016/j.jmaa.2015.04.085

and the harmonisable representation as an integral with respect to the Fourier transform of the white noise, see [7,15,23,30].

Istas [17] defined the fractional Brownian motion $B$ on a metric space by assuming that the square of its increment $B(x) - B(y)$ is normally distributed with the variance given by the $2H$-power of the metric distance between $x$ and $y$. The existence of fractional Brownian motions on the Euclidean sphere and on the hyperbolic space is verified for $H \in (0, \frac{1}{2}]$. These constructions have been extended to stable random fields in [18]. See also the recent monograph [7] for a number of results on general self-similar random fields.

Biermé et al. [3] considered a random field generated by a Poisson random measure on $\mathbb{R}^d \times [0, \infty)$ and proved that it shares the same covariance function (1) with the Lévy fBf. Such a field may be called a fractional Poisson field, noticing that other definitions of fractional Poisson fields are available in the literature, see e.g. [27] for the univariate case and [21] for a multivariate generalisation.

In this paper we introduce a generalisation of the fBf based on replacing the Euclidean norm in (1) with a non-Euclidean one. The space $\mathbb{R}^d$ with such norm is called the Minkowski space [33], so that we term our generalisation the Minkowski fractional Brownian field (MfBf). Section 2 introduces necessary concepts from convex geometry. In Section 3 we establish that the norms giving rise to valid covariance functions are generated by $L_p$-balls related to the isometric embeddability of the Minkowski space into $L_p([0, 1])$ for $p = 2H$. Furthermore, we derive several integral representations of the introduced random field. In addition to conventional integral representations based on integrating the white noise or its Fourier transform, we derive novel representations based on sums of series of Lévy fBf's and integrals of univariate fractional Brownian motions. Furthermore, we relate the ordering of expected supremum of MfBf with the Banach–Mazur distance between normed spaces. The key idea is the equivalence relation on the family of MfBf up to non-degenerate linear transformations of their arguments.

Section 4 introduces random fields based on Poisson point processes that share the covariance function with the MfBf for $H \in (0, \frac{1}{2})$. The construction follows the ideas from [3] and [34], and is also related to random balls models [6] and the studies of micropulses [26]. In difference to the previous works, we emphasise the role of the shape parameters of the corresponding fields. The main result provides a relationship between the shape parameters of the Poisson random field and its Gaussian counterpart. This relationship is given by the radial $p$th mean body transformation introduced in [12]. The convergence to the Brownian field with $H = \frac{1}{2}$ using different normalisations of the Poisson model is considered in Section 5. These results are new even in the case of Lévy fBf. The shape parameters are related by the polar projection body transform known from the convex geometry [31]. Finally Section 6 presents other constructions of the fractional Poisson fields that share the covariance structure with the MfBf.

## 2. Norms and star bodies

A closed bounded set $F$ in $\mathbb{R}^d$ is called a *star body* if for every $u \in F$ the interval $\{tu : \ 0 \le t < 1\}$ is contained in the interior of $F$ and the *Minkowski functional* (or the gauge function) of $F$ defined by

$$\|u\|_F = \inf\{s \ge 0 : \ u \in sF\}$$

is a continuous function of $u \in \mathbb{R}^d$. The set $F$ can be recovered from its Minkowski functional by

$$F = \{u : \ \|u\|_F \le 1\},$$

while the *radial function* $\rho_F(u) = \|u\|_F^{-1}$ provides the polar coordinate representation of the boundary of $F$ for $u$ from the unit *Euclidean sphere* $\mathbb{S}^{d-1}$. In the following we mainly consider origin-symmetric star-shaped sets and call them *centred* in this case. If the star body $F$ is centred and *convex*, then $\|u\|_F$ becomes a convex norm on $\mathbb{R}^d$ and $(\mathbb{R}^d, \|\cdot\|_F)$ is called a *Minkowski space*, see [33]. We also keep the same notation $\|u\|_F$ if