J. Math. Anal. Appl. 430 (2015) 279–295

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Discrete approximations on functional classes for the integrable nonlinear Schrödinger dynamical system: A symplectic finite-dimensional reduction approach

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ARTICLE INFO

Article history:
Received 19 February 2014
Available online 29 April 2015
Submitted by R. Popovych

Keywords:
Discretization schemes
Discrete nonlinear Schrödinger dynamical system
Lax type representation
Gradient-holonomic algorithm
Finite dimensional symplectic reduction approach

ABSTRACT

We investigate discretizations of the integrable discrete nonlinear Schrödinger dynamical system and related symplectic structures. We develop an effective scheme of invariant reducing the corresponding infinite system of ordinary differential equations to an equivalent finite system of ordinary differential equations with respect to the evolution parameter. We construct a finite set of recurrent algebraic regular relations allowing to generate solutions of the discrete nonlinear Schrödinger dynamical system and we discuss the related functional spaces of solutions. Finally, we discuss the Fourier transform approach to studying the solution set of the discrete nonlinear Schrödinger dynamical system and its functional–analytical aspects.

1. Introduction

With its origins going back several centuries, discrete analysis becomes now an increasingly central methodology for many mathematical problems related to discrete systems and algorithms, widely applied in modern science. Our theme, being related to studying integrable discretizations of nonlinear integrable dynamical systems and the limiting properties of their solution sets, is of deep interest in many branches of modern science and technology, especially in discrete mathematics, numerical analysis, statistics and probability theory as well as in electrical and electronic engineering. In fact, this topic belongs to a much more general realm of mathematics, namely to calculus, differential equations and differential geometry. Thereby, although the topic is discrete, our approach to treating this problem will be completely analytical.

In this work we will analyze the properties of discrete approximation for the nonlinear integrable Schrödinger (NLS) dynamical system on a functional manifold $\tilde{M} \subset L_2(\mathbb{R};\mathbb{C}^2)$:

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http://dx.doi.org/10.1016/j.jmaa.2015.04.084
0022-247X/

$$\left.\begin{aligned} \tfrac{d}{dt}\psi &= i\psi_{xx} - 2i\alpha\psi\psi\psi^*, \\ \tfrac{d}{dt}\psi^* &= -i\psi^*_{xx} + 2i\alpha\psi^*\psi\psi^* \end{aligned}\right\} := \tilde{K}[\psi, \psi^*], \tag{1.1}$$

where, by definition $(\psi, \psi^*)^\intercal \in \tilde{M}$, $\alpha \in \mathbb{R}$ is a constant, the subscript "x" means the partial derivative with respect to the independent variable $x \in \mathbb{R}$, $\tilde{K}: \tilde{M} \to T(\tilde{M})$ is the corresponding vector field on $\tilde{M}$, $t \in \mathbb{R}$ is the evolution parameter and, finally, $L_2(\mathbb{R}; \mathbb{C}^2)$ is the space of square-integrable functions $\mathbb{R} \to \mathbb{C}^2$. The system (1.1) possesses a Lax type representation (see [20]) and is Hamiltonian

$$\frac{d}{dt}(\psi, \psi^*)^\intercal = -\tilde{\theta}\, \operatorname{grad} \tilde{H}[\psi, \psi^*] = \tilde{K}[\psi, \psi^*] \tag{1.2}$$

with respect to the canonical Poisson structure $\tilde{\theta}$ and the Hamiltonian function $\tilde{H}$, where

$$\tilde{\theta} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.3}$$

is a nondegenerate mapping $\tilde{\theta}: T^*(\tilde{M}) \to T(\tilde{M})$ on the smooth functional manifold $\tilde{M}$, and

$$\tilde{H} := \frac{1}{2} \int\limits_{\mathbb{R}} dx \left[\psi\psi^*_{xx} + \psi_{xx}\psi^* - 2\alpha(\psi^*\psi)^2\right] \tag{1.4}$$

is a smooth mapping $\tilde{H}: \tilde{M} \to \mathbb{C}$. The corresponding symplectic structure [4,5,8,7] for the Poisson operator (1.3) is defined by

$$\begin{aligned} \tilde{\omega}^{(2)} &:= -\frac{i}{2} \int\limits_{\mathbb{R}} dx\, \langle (d\psi, d\psi^*)^\intercal, \wedge \tilde{\theta}^{-1}(d\psi, d\psi^*)^\intercal \rangle = \\ &= -i \int\limits_{\mathbb{R}} dx\, \left[d\psi^*(x) \wedge d\psi(x)\right], \end{aligned} \tag{1.5}$$

which is a nondegenerate and closed 2-form on the functional manifold $\tilde{M}$.

The simplest spatial discretizations of the dynamical system (1.1), defined by

$$\begin{aligned} \frac{d}{dt}\psi_n &= \frac{i}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) - 2i\alpha\psi_n\psi_n\psi^*_n, \\ \frac{d}{dt}\psi^*_n &= -\frac{i}{h^2}(\psi^*_{n+1} - 2\psi^*_n + \psi^*_{n-1}) + 2i\alpha\psi^*_n\psi_n\psi^*_n \end{aligned} \tag{1.6}$$

and

$$\left.\begin{aligned} \tfrac{d}{dt}\psi_n &= \tfrac{i}{h^2}(\psi_{n+1} - 2\psi_n + \psi_{n-1}) - i\alpha(\psi_{n+1} + \psi_{n-1})\psi_n\psi^*_n, \\ \tfrac{d}{dt}\psi^*_n &= -\tfrac{i}{h^2}(\psi^*_{n+1} - 2\psi^*_n + \psi^*_{n-1}) + i\alpha(\psi^*_{n+1} + \psi^*_{n-1})\psi_n\psi^*_n, \end{aligned}\right\} := K[\psi_n, \psi^*_n], \tag{1.7}$$

are flows on some "discrete" submanifold M_h, where, by definition, $\{(\psi_n, \psi^*_n)^\intercal \in \mathbb{C}^2: n \in \mathbb{Z}\} \subset M_h \subset l_{h,2}(\mathbb{Z}; \mathbb{C}^2)$. Here

$$l_{h,2}(\mathbb{Z}; \mathbb{C}^2) := \{(\psi_n, \psi^*_n)^\intercal,\ n \in \mathbb{Z} : \sum_{n\in\mathbb{Z}} |\psi_n|^2 h < \infty\} \tag{1.8}$$

is the space of square-integrable complex vector-valued sequences with the invariant positive wage $h > 0$, and $K: M_h \to T(M_h)$ is the corresponding vector field on M_h.