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Quasi-linear systems of PDE of first order with Cauchy data of higher codimensions ☆

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A B S T R A C T

In this paper we discuss the local solvability of Cauchy problem for quasi-linear partial differential equations of first order. By using the classical method of characteristics we describe the non-uniqueness or the degree of freedom for solutions and also decide the conditions for the existence and the uniqueness of solutions for overdetermined systems of quasi-linear PDEs of first order.

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0. Introduction

Consider the quasi-linear partial differential equation (PDE)

$$\sum_{\lambda=1}^{n} a^\lambda(x,u)\frac{\partial u}{\partial x^\lambda} = b(x,u), \tag{0.1}$$

for one real-valued function u in n real variables $x = (x^1, \dots, x^n)$. We assume that $a^1, \dots, a^n, b$ are continuously differentiable ($\mathcal{C}^1$-class).

By a Cauchy data of codimension 1 we mean a pair

$$\Gamma(s) = (x(s), u(s)), \quad \text{where } s = (s^1, \dots, s^{n-1}), \tag{0.2}$$

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of a hypersurface $x(s)$ in the space of x and $u(s)$ the value of u assigned to $x(s)$. The Cauchy data $\Gamma(s)$ is called *non-characteristic* if

$$\det \begin{bmatrix} \frac{\partial x^1}{\partial s^1} & \cdots & \frac{\partial x^1}{\partial s^{n-1}} & a^1 \\ \vdots & & \vdots & \vdots \\ \frac{\partial x^n}{\partial s^1} & \cdots & \frac{\partial x^n}{\partial s^{n-1}} & a^n \end{bmatrix} \neq 0 \quad \text{on } \Gamma(s).$$

The Cauchy problem is to find a solution $u = u(x)$ of (0.1) subject to the Cauchy data (0.2). The Cauchy problem has a unique solution if the Cauchy data is non-characteristic. The classical *method of characteristics* was originated from the work of G. Monge [13] to solve the Cauchy problem.

The vector field

$$X = \sum_{\lambda=1}^{n} a^\lambda \frac{\partial}{\partial x^\lambda} + b\frac{\partial}{\partial u} \tag{0.3}$$

on $\mathbb{R}^{n+1} = \{(x^1, \ldots, x^n, u)\}$ is called the *characteristic vector field* associated with (0.1). Geometrically, (0.1) means that a function $u = f(x)$ is a solution if and only if X is tangent to the graph $(x, f(x))$, that is, $\phi(x, u) := u - f(x)$ satisfies

$$X\phi = 0, \qquad \text{on } \phi = 0.$$

Let

$$\gamma_s(t) = (x(s,t), u(s,t)), \quad -\epsilon < t < \epsilon$$

be the integral curve of X subject to the initial condition $\gamma_s(0) = \Gamma(s)$, for each s. Then

$$(s,t) \mapsto (x(s,t), u(s,t)) \tag{0.4}$$

is a hypersurface in the $(n+1)$-space of (x, u) and is the graph of the unique solution of (0.1) subject to the initial condition (0.2). The existence of the mapping (0.4) is a consequence of the fundamental theorem of ODE, see [2,11]. If the Cauchy data $\Gamma(s)$ is $\mathcal{C}^1$, then the solution is $\mathcal{C}^1$. In practice, one finds the solution of the Cauchy problem (0.1)–(0.2) by using the first integrals of the characteristic vector field X. A $\mathcal{C}^1$ function ϕ is a *first integral* of X if ϕ is constant along each integral curve, that is,

$$X\phi = 0.$$

The first purpose of this paper is to describe the non-uniqueness of the solutions to the Cauchy problems with Cauchy data of *higher codimensions* and to present an algorithmic method of finding local solutions. In Section 2, we define the non-charactericity of Cauchy data of higher codimensions and decide the conditions for the uniqueness of solutions. We use the classical method of characteristics and the 'flow box' theorem as in [14,10] and the theory of first integrals as in [4], and thus our arguments are ultimately based on the fundamental theorems of ODE [1]. This algorithmic method always provides an implicit solution if it exists.

The second purpose of this paper is to study the same problem for the systems. We discuss first the Cauchy problem for a system of p $(1 \leq p \leq n)$ quasi-linear partial differential equations for one real unknown function u in n real variables $x = (x^1, \ldots, x^n)$:

$$\sum_{\lambda=1}^{n} a_j^\lambda(x,u)\frac{\partial u}{\partial x^\lambda} = b_j(x,u), \quad j = 1, \ldots, p, \tag{0.5}$$

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