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Upper bounds on the eigenvalue ratio for vibrating strings

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ABSTRACT

We apply a commutation formula of Deift to study the eigenvalue ratios of vibrating strings with fixed endpoints. Some new results on the upper bounds of the eigenvalue ratio are established.

1. Introduction

Consider two bounded linear operators A and B on a Hilbert space. One shows, by multiplying out, that for each $\lambda \neq 0$ in the resolvent set of AB, the operator

$$\lambda^{-1}\left[B\left(AB-\lambda\right)^{-1}A-I\right]$$

is a two sided inverse of $BA-\lambda$, so that λ also belongs to the resolvent set of BA. It follows, by symmetry, that AB and BA have the same spectrum away from zero:

$$\sigma\left(AB\right)\setminus\{0\}=\sigma\left(BA\right)\setminus\{0\}. \tag{1.1}$$

Deift [4] gave a proof of an extended version of the commutation formula (1.1) which includes the case where A is a closed, densely defined operator and $B=A^*$, its adjoint.

Theorem 1.1. *If A is a closed, densely defined linear operator on a Hilbert space with adjoint A^*, then AA^* and A^*A have the same spectrum away from zero. Moreover, if $\lambda \neq 0$ is an eigenvalue of AA^**

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(resp. A^*A), then λ is also an eigenvalue of A^*A (resp. AA^*), and $\mathrm{N}(AA^* - \lambda)$ and $\mathrm{N}(A^*A - \lambda)$ have the same dimension, where N stands for the nullspace.

Commutation formula is a useful tool in finding bounds on eigenvalue ratios. This has been emphasized by Ashbaugh and Benguria in [1] and [3]. For example, it was shown using the commutation formula that $\lambda_2/\lambda_1 \leq 4$ for the ratio of the first two eigenvalues of one-dimensional Schrödinger operators with nonnegative potentials, and equality holds if and only if the potential vanishes identically [1].

In this paper, we want to apply Theorem 1.1 to study the eigenvalue ratios of vibrating strings with fixed endpoints. We consider the following eigenvalue problem:

$$\begin{cases} u''(x) + \lambda\rho(x)u(x) = 0 & \text{in } (a,b) \\ u(a) = u(b) = 0, \end{cases} \tag{1.2}$$

where $\rho(x)$ is a positive continuous function and represents the mass density. As is well known, the eigenvalues of (1.2) form a strictly increasing sequence of positive numbers:

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots.$$

We shall be concerned with the ratio of the first two eigenvalues of (1.2). Our main result is:

Theorem 1.2. *Let $\alpha > 1/2$. Suppose that ρ is twice differentiable and that either*

$$\rho(x)\rho''(x) \geq \frac{1}{(\alpha-2)^2}\left[\alpha^2 - 8\alpha + 6 + \sqrt{\alpha(\alpha+4)(2\alpha-1)}\right][\rho'(x)]^2 \tag{1.3}$$

or

$$\rho(x)\rho''(x) \leq \frac{1}{(\alpha-2)^2}\left[\alpha^2 - 8\alpha + 6 - \sqrt{\alpha(\alpha+4)(2\alpha-1)}\right][\rho'(x)]^2 \tag{1.4}$$

for all x in (a,b). Then

$$\frac{\lambda_2}{\lambda_1} \leq \frac{\alpha(\alpha+4)}{(2\alpha-1)}. \tag{1.5}$$

Remark 1.1. (a) The minimum of the right-hand side of (1.5) for $\alpha > 1/2$ occurs at $\alpha = 2$ and is 4. When $\alpha = 2$, the condition (1.3) is to be understood as follows:

$$\begin{aligned} \rho\rho'' &\geq \lim_{\alpha\to 2}\frac{1}{(\alpha-2)^2}\left[\alpha^2 - 8\alpha + 6 + \sqrt{\alpha(\alpha+4)(2\alpha-1)}\right](\rho')^2 \\ &= \frac{5}{4}(\rho')^2. \end{aligned}$$

Under this condition, the eigenvalue ratio satisfies $\lambda_2/\lambda_1 \leq 4$.

(b) It is a result of Ashbaugh and Benguria [2] that if $\rho^{-1/4}$ is concave, then $\lambda_n/\lambda_1 \leq n^2$. We note that if ρ is twice differentiable, then the concavity of $\rho^{-1/4}$ is equivalent to $\rho\rho'' \geq \frac{5}{4}(\rho')^2$.

Remark 1.2. An elementary calculation shows that

$$\alpha^2 - 8\alpha + 6 + \sqrt{\alpha(\alpha+4)(2\alpha-1)} \geq 0$$

for any $\alpha > 1/2$. Thus, if ρ satisfies the condition (1.3), then ρ must be convex.