J. Math. Anal. Appl. 430 (2015) 417–427

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Continuous and localized Riesz bases for L^2 spaces defined by Muckenhoupt weights

Hugo Aimar [a], Wilfredo A. Ramos [b,c,*]

[a] Instituto de Matemática Aplicada del Litoral, IMAL (UNL-CONICET), CCT CONICET Santa Fe, Predio "Dr. Alberto Cassano", Colectora Ruta Nac. 168, Paraje El Pozo, 3000 Santa Fe, Argentina
[b] Instituto de Matemática Aplicada del Litoral, IMAL (UNL-CONICET), Argentina
[c] Departamento de Matemática, Facultad de Ciencias Exactas Naturales y Agrimensura, Universidad Nacional del Nordeste, Argentina

ARTICLE INFO

Article history:
Received 14 January 2015
Available online 6 May 2015
Submitted by R.H. Torres

Keywords:
Riesz bases
Haar wavelets
Basis perturbations
Muckenhoupt weights
Cotlar's Lemma

ABSTRACT

Let w be an A_∞-Muckenhoupt weight in $\mathbb{R}$. Let $L^2(wdx)$ denote the space of square integrable real functions with the measure $w(x)dx$ and the weighted scalar product $\langle f, g\rangle_w = \int_{\mathbb{R}} fg\ wdx$. By regularization of an unbalanced Haar system in $L^2(wdx)$ we construct absolutely continuous Riesz bases with supports as close to the dyadic intervals as desired. Also the Riesz bounds can be chosen as close to 1 as desired. The main tool used in the proof is Cotlar's Lemma.

1. Introduction and statement of the main result

A sequence $\{f_k, k \in \mathbb{Z}\}$ in a Hilbert space H is said to be a Bessel sequence with bound B if the inequality

$$\sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \le B \,\|f\|_H^2$$

holds for every $f \in H$. If $\{f_k, k \in \mathbb{Z}\}$ is a Bessel sequence with bound B and $\{e_k, k \in \mathbb{Z}\}$ is an orthonormal basis for the separable Hilbert space H, then the operator T on H defined by

$$Tf := \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle e_k$$

is bounded on H with bound $\sqrt{B}$. Conversely if T is bounded on H, then $\{f_k,\ k \in \mathbb{Z}\}$ is a Bessel sequence with bound $\|T\|^2$.

* Corresponding author.
E-mail addresses: haimar@santafe-conicet.gov.ar (H. Aimar), wramos@santafe-conicet.gov.ar (W.A. Ramos).

http://dx.doi.org/10.1016/j.jmaa.2015.05.003
0022-247X/

When $\{f_k,\ k \in \mathbb{Z}\}$ itself is an orthonormal basis and $e_k = f_k$, T is the identity. Of particular interest is the case of $H = L^2$ when the Bessel system and the orthonormal basis are built on scaling and translations of the underlying space. In such cases the operator T has a natural decomposition as $T = \sum_{j\in\mathbb{Z}} T_j$. Sometimes the orthonormal basis can be chosen in such a way that the T_j's become almost orthogonal in the sense of Cotlar. We aim to use Cotlar's Lemma to produce smooth and localized Riesz bases for $L^2(\mathbb{R}, wdx)$ when w is a Muckenhoupt weight.

To introduce the problem let us start by some simple illustrations. Let ψ be a Daubechies compactly supported wavelet in $\mathbb{R}$. Assume that supp$\psi \subset [-N, N]$. The system $\{\tilde{\psi}^j_k(x) = 2^{\frac{j}{2}}\psi(2^j x^3 - k) :\ j, k \in \mathbb{Z}\}$ is a compactly supported orthonormal basis for $L^2(\mathbb{R}, 3x^2dx)$. More generally if $w(x)$ is a non-negative locally integrable function in $\mathbb{R}$ and $W(x) = \int_0^x w(y)dy$, then the system $\overline{\psi}^j_k(x) = 2^{\frac{j}{2}}\psi(2^j W(x) - k)$ is an orthonormal basis for $L^2(wdx)$. In fact, changing variables

$$\begin{aligned}\int_{\mathbb{R}} \overline{\psi}^j_k(x)\overline{\psi}^l_m(x)w(x)dx &= 2^{\frac{l+j}{2}} \int_{\mathbb{R}} \psi(2^j W(x) - k)\psi(2^l W(x) - m)w(x)dx \\ &= \int_{\mathbb{R}} \psi^j_k(z)\psi^l_m(z)dz\end{aligned}$$

and we have the orthonormality of the system $\{\overline{\psi}^j_k : j \in \mathbb{Z},\ k \in \mathbb{Z}\}$ in $L^2(\mathbb{R}, wdx)$. As it is easy to verify in the case of $w(x) = 3x^2$, for j fixed the length of the supports of $\overline{\psi}^j_k$ tend to zero as $|k| \to +\infty$. On the other hand for $k = 0$ the scaling parameter is $2^{-\frac{1}{3}}$.

Notice also that if w is bounded above and below by positive constants the sequence $\overline{\psi}^j_k$ is an orthonormal basis for $L^2(wdx)$ with a metric control on the sizes of the supports provided by the scale.

A Riesz basis in $L^2(wdx)$ is a Schauder basis $\{f_k\}$ such that there exist two constants A and B called the Riesz bounds of $\{f_k\}$ for which

$$A\sum |c_k|^2 \le \left\|\sum c_k f_k\right\|^2_{L^2(wdx)} \le B\sum |c_k|^2$$

for every $\{c_k\}$ in $l^2(\mathbb{R})$, the space of square summable sequences of real numbers. In this note we aim to give sufficient conditions on a weight w defined on $\mathbb{R}$ more general than $0 < c_1 \le w(x) \le c_2 < \infty$, in order to construct, for every $\delta > 0$, a system $\Psi = \{\psi_I(x),\ I \in \mathcal{D}\}$ ($\mathcal{D}$ are the dyadic intervals in $\mathbb{R}$) with the following properties,

(i) Ψ is a Riesz basis for $L^2(wdx)$ with bounds $(1 - \delta)$ and $(1 + \delta)$,
(ii) each ψ^j_k is absolutely continuous,
(iii) for each I, ψ_I is supported on a neighborhood I^ϵ of I such that

$$0 < \frac{|I^\epsilon|}{|I|} - 1 < \delta.$$

As we have shown in the above example with $w(x) = 3x^2$, we have that $\{\overline{\psi}^j_k\}$ satisfies (i) and (ii) but not (iii).

An orthonormal basis in $L^2(\mathbb{R}, wdx)$ satisfying (iii) but not (ii) when w is locally integrable is the following unbalanced version of the Haar system (see [12]). Let $\mathcal{D} = \cup_{j\in\mathbb{Z}}\mathcal{D}^j$ be the family of standard dyadic intervals in $\mathbb{R}$. Each I in $\mathcal{D}^j$ takes the form $I = [k2^{-j},\ (k + 1)2^{-j})$ for same integer k. For $I \in \mathcal{D}^j$ we have that $|I| = 2^{-j}$. We shall frequently use a_I and b_I to denote the left and right points of I respectively, for each $I \in \mathcal{D}$, define