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A typical copula is singular

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ABSTRACT

We present Baire category results in the class of bivariate copulas (or, equivalently, doubly stochastic probability measures) endowed with two different metrics under which the space is complete. Main content of the paper is that, in the sense of Baire categories with respect to the topology induced by the uniform metric, the family of absolutely continuous copulas is of first category, whereas the family of purely singular copulas is co-meager and, hence, of second category. Moreover, several other popular dense sub-classes of copulas are considered, like shuffles of Min and checkerboard copulas.

1. Introduction

Starting with the seminal paper by A. Sklar [28] copulas have gathered a lot of interest because of their relevance to model stochastic dependence and, as such, they constitute a relevant tool in applied probability and statistics. The interested reader may refer, for instance, to the recent monographs [5,8,13–15,20,23,27] and the references therein.

The growing interest of copulas has also generated a number of different ways to provide families of copulas (e.g., shuffles, Bernstein, etc.) that are dense (in a given topology) in the class of copulas. However, little attention has been devoted to the question whether such families are big (or small) in a specific sense, i.e. whether they cover a sufficiently large spectrum of the copula space.

For a given metric space, topology offers a natural way of distinguishing small and big sets through Baire categories (see, e.g. [24]). Roughly speaking, a subset of a metric space (S, d) is considered to be "small" if it is *nowhere dense*, i.e. if it is not dense in any non-degenerated open ball $B_r(x, d)$ of radius $r > 0$ (equivalently, if its closure has empty interior). A set $A \subseteq S$ is called *meager* or of *first category* in (S, d) if it is expressible as a countable union of nowhere dense sets. A is called of *second category* if it is not meager.

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Finally, A is called *co-meager* if $A^c = S \setminus A$ is meager. Loosely speaking, we will also refer to the elements of a co-meager set as *typical* and to the elements of a meager set as *atypical* in S.

In statistical theory, as stressed in [25], many asymptotic results do not hold for all underlying distributions $P \in \mathscr{P}$, but for all $P \in \mathscr{P} \setminus \mathscr{D}$, where $\mathscr{D}$ is a "small" compared to $\mathscr{P}$. In this context, small set is usually intended in the sense of Baire categories. For instance, in [17] the authors used Baire category arguments to motivate their interest in developing kernel density estimators that work in a wider class of densities. Analogously, in [25] (see also [26]) some bootstrap procedures are shown to work outside a set of first category (in the class of all possible underlying probability distributions).

A first study of Baire category results for copulas was presented in [18], where it is proved that the set of copula models that are dynamically consistent (with respect to a specific joint default model) and satisfy some technical regularity conditions, is a set of the first category in the Baire sense in a certain space of copulas with finitely many parameters. Recently, category results for exchangeable copulas and related families (associative, Archimedean) have been investigated in [6], motivated by an open problem proposed in [1].

Following this general framework, here we concentrate on the study of classes of copulas characterized by their measure-theoretic properties, namely absolutely continuous and purely singular copulas.

The main results are the following: the class of absolutely continuous copulas is of first category, while the class of (purely) singular copulas is of second category in the space $(\mathscr{C}, d_\infty)$ of all two-dimensional copulas with the uniform metric. Related results are also established for the strong metric D_1 introduced in [29] and special sub-classes of copulas like checkerboard copulas and shuffles of Min. For the sake of simplicity, all the results will be presented in the bivariate case. However, they can be also formulated in d-dimension ($d \geq 3$) with simple and obvious modifications.

2. Notation and preliminaries

In the sequel Ω will denote a square of the form $\Omega := [e, e+\delta] \times [f, f+\delta] \subseteq [0,1]^2$ with $\delta > 0$. $\mathscr{B}(\Omega)$ will denote the Borel σ-field in Ω, $\mathscr{M}(\Omega)$ the family of all finite (positive) measures on $(\Omega, \mathscr{B}(\Omega))$, and $\mathscr{M}_m(\Omega)$ the family of all $\mu \in \mathscr{M}(\Omega)$ fulfilling $\mu(\Omega) = m$, whereby $m \in (0, \infty)$. For every $\mu \in \mathscr{M}(\Omega)$ the corresponding measure-generating function (see, e.g., [2]) will be denoted by F_μ and is defined, for every $(x, y) \in \Omega$, by $F_\mu(x, y) = \mu([e, x] \times [f, y])$. We will use the symbols $\mathscr{F}(\Omega)$ and $\mathscr{F}_m(\Omega)$ for the families of the measure-generating functions corresponding to elements of $\mathscr{M}(\Omega)$ and $\mathscr{M}_m(\Omega)$ respectively. Given $\mu \in \mathscr{M}(\Omega)$ we will write $\mu = \mu_s + \mu_a$ for the Lebesgue decomposition of μ, where μ_a (respectively, μ_s) is absolutely continuous with respect to the Lebesgue measure λ_2; in particular, k_μ will denote the density of μ_a. For the sake of simplicity we will also write k_F instead of k_μ if F is the measure-generating function corresponding to μ. For every $L \in \mathbb{N}$ and $\mu \in \mathscr{M}(\Omega)$ the set B^L_μ is formed by all points in Ω such that the density of μ_a is upper bounded by L and strictly positive, namely

$$B^L_\mu := \{(x, y) \in \Omega : 0 < k_\mu(x, y) \leq L\}. \tag{2.1}$$

The symbol $\mathscr{C}$ will denote the family of all two-dimensional copulas, $\mathscr{P}_\mathscr{C} \subset \mathscr{M}_1([0,1]^2)$ the family of all doubly stochastic measures. It is well-known that there is a one-to-one correspondence between $\mathscr{C}$ and $\mathscr{P}_\mathscr{C}$. Moreover, $\mathscr{C}_{abs}$ (respectively, $\mathscr{C}_{sing}$) denotes the subclass of all absolutely continuous (respectively, singular) copulas, i.e. those copulas whose induced measure is absolutely continuous (respectively, singular) with respect to λ_2. Notice that $\mathscr{C}$ equipped with the d_∞ metric (respectively, D_1 metric by [29]) is complete and, hence, is a Baire space (see, e.g., [29]).

In the sequel we will also work with slightly generalized checkerboard-like constructions induced by so-called transformation matrices (see [11,29]): An $n \times m$ matrix $\tau = (\tau_{ij}) \in [0,1]^{n \times m}$ is called *transformation matrix* if no row or column has all entries 0 and $\sum_{i,j} \tau_{i,j} = 1$. We will let $\mathscr{T}$ denote the family of all