J. Math. Anal. Appl. 430 (2015) 549–571

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Identifying weak foci and centers in the Maxwell–Bloch system

Lingling Liu [a,b], O. Ozgur Aybar [c], Valery G. Romanovski [d,e], Weinian Zhang [b,*]

[a] *School of Sciences, Southwest Petroleum University, Chengdu, Sichuan 610500, PR China*
[b] *Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, PR China*
[c] *Piri Reis University, 34940, Tuzla, Istanbul, Turkey*
[d] *Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SI-2000, Slovenia*
[e] *Faculty of Natural Science and Mathematics, University of Maribor, SI-2000, Maribor, Slovenia*

ARTICLE INFO

Article history:
Received 23 July 2014
Available online 7 May 2015
Submitted by J. Shi

Keywords:
Maxwell–Bloch system
Center-focus
Algebraic variety
Invariant algebraic surface
Singular center

ABSTRACT

In this paper we identify weak foci and centers in the Maxwell–Bloch system, a three dimensional quadratic system whose three equilibria are all possible to be of center-focus type. Applying irreducible decomposition and the isolation of real roots in computation of algebraic varieties of Lyapunov quantities on an approximated center manifold, we prove that at most 6 limit cycles arise from Hopf bifurcations and give conditions for exact number of limit cycles near each weak focus. Further, applying algorithms of computational commutative algebra we find Darboux polynomials and give some center manifolds in closed form globally, on which we identify equilibria to be centers or singular centers by integrability and time-reversibility on a center manifold. We prove that those centers are of at most second order.

1. Introduction

The description of the interaction between laser light and a material sample composed of two-level atoms begins with Maxwell equations of the electric field and Schrödinger equations for the probability amplitudes of the atomic levels [10,16]. In 1985, coupling the Maxwell equations with the Bloch equation (a linear Schrödinger like equation describing the evolution of atoms resonantly coupled to the laser field), F.T. Arecchi [1] proposed the 3-dimensional quadratic differential system

$$\begin{cases} \dot{E} = -k\,E + g\,P, \\ \dot{P} = -\gamma_\perp\,P + g\,E\Delta, \\ \dot{\Delta} = -\gamma_\parallel\,(\Delta - \delta_0) - 4g\,PE, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail address: matzwn@126.com (W. Zhang).

http://dx.doi.org/10.1016/j.jmaa.2015.05.007

called the Maxwell–Bloch system, where E is the coupling of the fundamental cavity mode, P is the collective atomic polarization, Δ represents the population inversion, k, $\gamma_\perp$, $\gamma_\parallel$ are the loss rates of field, polarization and population respectively, g is a coupling constant ($g \neq 0$) and δ_0 is the population inversion which would be established by the pump mechanism in the atomic medium, in the absence of coupling. As indicated in [1,15,17], the Maxwell–Bloch system describes Type I laser (He-Ne), Type II laser (Ruby and CO_2) and Type III laser (far infrared) when $\gamma_\perp \approx \gamma_\parallel \gg k$, when $\gamma_\perp \gg \gamma_\parallel \approx k$ and when Δ_0 is large enough, respectively. Numerical simulations [1] show that Type I laser and Type II laser are both stable but Type III laser is unstable.

A few papers [15,23] were published to discuss this 3-dimensional system in mathematical aspect. In 1998 Puta [23] considered the integrability in the special case that $k = \gamma_\perp = \gamma_\parallel = 0$. In 2010 Hacinliyan, Kusbeyzi and Aybar [15] indicated that the pair of equilibria which are $\mathcal{C}_2$-symmetric with respect to the Δ-axis are both of center-focus type and showed the rise of a stable limit cycle from a Hopf bifurcation, verifying that a pair of conjugate complex eigenvalues crosses the imaginary axis and varying the parameter g to obtain a negative first Lyapunov coefficient numerically. However, the work of [15] neither answers the order of weak foci nor identifies a center. Actually, it causes great complexity in computation to identify weak foci and centers for 3-dimensional systems [9,14,19–21]. Concerning weak foci, a natural idea is to compute Lyapunov quantities for the restriction of the 3-dimensional system to an approximated center manifold, but the approximation to the center manifold makes the complicated computation of algebraic varieties of Lyapunov quantities more difficult. Concerning centers, those criteria for centers in planar systems, seen in [22,24] for time-reversibility and integrability, are not effective on an approximated center manifold. In [20] such identifications for weak focus and center were completed for the generalized Lorenz system, a 3-dimensional system, by approximating a local center manifold and finding the closed form of a global center manifold respectively. However, system (1.1) is quite different because it is only for $\delta_0 = (k+1)(\gamma_\perp+1)$ that system (1.1) can be transformed into the generalized Lorenz form.

In this paper we identify weak foci and centers for the Maxwell–Bloch system (1.1) qualitatively. With a time rescaling $\tau_1 = gt$, system (1.1) can be simplified as the following equivalent form

$$\begin{cases} \dot{x}_1 = -a\,x_1 + x_2, \\ \dot{x}_2 = -b\,x_2 + x_1x_3, \\ \dot{x}_3 = -c\,(x_3 - \delta_0) - 4\,x_1x_2, \end{cases} \tag{1.2}$$

where x_1, x_2, x_3 simply present E, P, Δ respectively and $a = g^{-1}k$, $b = g^{-1}\gamma_\perp$, $c = g^{-1}\gamma_\parallel$, the ratios of the loss rates of field k, polarization $\gamma_\perp$ and population $\gamma_\parallel$ respectively to the coupling constant. The three equilibria E_0 and $E_\pm$ are all possible to be of center-focus type. We prove that the system can totally produce at most 6 limit cycles from those weak foci. Applying irreducible decomposition and the isolation of real roots in computation of algebraic varieties of Lyapunov quantities on approximated center manifolds, we give conditions for exact number of limit cycles near each weak focus. Further, applying algorithms of computational commutative algebra, we find Darboux factors in polynomial form, which give some center manifolds in closed form globally. We identify equilibria to be centers or singular centers by proving integrability and time-reversibility on a center manifold. We prove that $E_\pm$ are both rough centers but E_0 is a center of at most order two.

2. Weak foci

As known in [15], the authors give the qualitative properties of all equilibria for system (1.1). The reduced system (1.2) containing less parameters helps us in latter computation of center manifolds, normal forms and those determining quantities. Clearly, for $c = 0$ system (1.2) has a singular line $\{(x_1, x_2, x_3)|x_1 = 0, x_2 = 0\}$.