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Lattice points in the 3-dimensional torus

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We prove the exponent 4/3 for the lattice point discrepancy of a torus in $\mathbb{R}^3$ (generated by the rotation of a circle around the z axis). The exponent comes from a diagonal term and it seems a natural limit for any approach based solely on classical methods of exponential sums. The result extends to other solids in $\mathbb{R}^3$ related to the torus.

1. Introduction

Consider a fixed compact solid body $\mathbb{B} \subset \mathbb{R}^3$. The study of

$$\#\{\vec{n} \in \mathbb{Z}^3 \;:\; R^{-1}\vec{n} \in \mathbb{B}\}, \qquad R > 1,$$

is a basic problem in lattice point theory. Under very general regularity conditions, this quantity is well approximated by $|\mathbb{B}|R^3$ where $|\mathbb{B}|$ denotes the volume of $\mathbb{B}$. Assuming regularity and convexity, meaning positive Gaussian curvature, the error in this approximation, the so-called lattice point discrepancy, is asymptotically smaller than the scaled area of the boundary, namely it is $O(R^\gamma)$ for some $\gamma < 2$. Many authors have devoted their efforts to give a partial answer to the natural question of determining the infimum of the valid values of γ. Commonly these results depend on subtle exponential sum techniques. The question remains open even for simple bodies like the sphere (see the survey [10] and the new results in [6]).

The non-convex case has also attracted the interest of researchers especially in the last decade (see for instance [17,13,12,15,16,7]). A notable difference is that sometimes the main term has to be complemented with a secondary main term coming, in some way, from the points of vanishing curvature. Note for instance

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that if $\mathbb{B}$ is the cube $[-1,1]^3$ with smoothed corners and edges (preserving the flatness of the faces) then $\#\{\vec{n}\in\mathbb{Z}^3 \,:\, R^{-1}\vec{n}\in\mathbb{B}\}$ counts more than R^2 on the boundary when $R\in\mathbb{Z}^+$.

A particularly symmetric example, considered in [15], is the solid torus

$$\mathbb{T} = \Big\{(x,y,z)\in\mathbb{R}^3 \,:\, \big(\rho' - \sqrt{x^2+y^2}\,\big)^2 + z^2 \le \rho^2\Big\} \tag{1}$$

where $0<\rho<\rho'$ are fixed constants. Its volume is $2\pi^2\rho^2\rho'$ but the natural main term to approximate the number of lattice points in the R-scaled torus is given by

$$\mathcal{M}(R) = 2\pi^2\rho^2\rho' R^3 + 4\pi\rho\rho' R^2 \sum_{n=1}^{\infty} \frac{J_1(2\pi\rho R n)}{n} \tag{2}$$

where J_1 is the Bessel function. The second main term has an oscillatory character and it is well approximated by $R^{3/2}$ times a ρ^{-1}-periodic function. In particular, when $R\in\rho^{-1}\mathbb{Z}^+$, up to a negligible error term, (2) can be substituted by $2\pi^2\rho^2\rho' R^3 + C\rho^{1/2}\rho' R^{3/2}$ with C a certain constant.

Our main result bounds the lattice point discrepancy when $\mathcal{M}(R)$ is taken as the main term.

Theorem 1.1. *With the notation of (1) and (2), consider*

$$\mathcal{N}(R) = \#\{\vec{n}\in\mathbb{Z}^3 \,:\, R^{-1}\vec{n}\in\mathbb{T}\}, \qquad \textit{and} \qquad \mathcal{E}(R) = \mathcal{N}(R) - \mathcal{M}(R).$$

Then we have

$$\mathcal{E}(R) = O\big(R^{4/3+\epsilon}\big) \qquad \textit{for every} \quad \epsilon > 0.$$

The previously best known result is due to W.G. Nowak [15] who obtained $\mathcal{E}(R) = O\big(R^{11/8+\epsilon}\big)$. He wrote $\mathcal{M}(R)$ as a series depending on elementary functions but it is equivalent to our statement after substituting the asymptotic formula for J_1.

The exponent 4/3 is better than the best known result for general convex bodies of rotation. This should be stressed because in principle the non-convex case is more difficult from the analytic point of view. The exponent comes from a diagonal term and then it seems unlikely to be improved in the context of the classical approaches based on exponential sums. If one could count with precision (beyond the limits of the harmonic analysis) points close to the boundary of the scaled theory then one could parallel the arguments of [4] (improved in [8]) or [3] to go beyond 4/3. The multiplicative harmonics (Dirichlet characters) and the automorphic harmonics employed in these papers apparently cannot be adapted to the torus.

The structure of the paper is as follows: In Section 2 we apply the Poisson summation formula to obtain formulas for $\mathcal{N}(R)$ and $\mathcal{E}(R)$. In Section 3 an application of the stationary phase principle allows to interpret the formula for $\mathcal{E}(R)$ as an exponential sum. In [15] this aim is reached indirectly thanks to the truncated Hardy–Voronoï formula for the circle problem [9] (the circles appear because each horizontal section of $\mathbb{T}$ is a corona). The advantages of our approach are that it is applicable to other problems, it is self-contained and it reveals the main term in a more transparent way. We devote Section 4 to estimate some exponential sums. Part of the work is done in [2] and [4]. In Section 5 we combine the results of the previous sections to get Theorem 1.1. Finally, in Section 6 we consider some extensions of our method.

In the following sections we assume $\rho' = 1$. It can be done without loss of generality because $\mathcal{N}(R)$ is clearly invariant by $(R,\rho,\rho')\mapsto(\lambda^{-1}R,\lambda\rho,\lambda\rho')$. We use the standard notation $e(x)=e^{2\pi i x}$ and ϵ denotes in this paper an arbitrarily small quantity, not necessarily the same each time. The constants involved in O and $\ll$ (notations that we consider equivalent) may depend on ϵ and on ρ.