J. Math. Anal. Appl. 429 (2015) 758–773

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Analyzing displacement term's memory effect in a nonlinear boundary value problem to prove chaotic vibration of the wave equation

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ARTICLE INFO

Article history:
Received 17 March 2015
Available online 15 April 2015
Submitted by Goong Chen

Keywords:
Chaotic wave vibrations
Memory effects
Time-delay displacement
Method of characteristics

ABSTRACT

Consider a nonlinear boundary value problem for the one-dimensional wave equation with the left-end boundary condition containing an extra linear displacement feedback term, which induces a memory effect. Technical difficulty arises as to how to determine chaotic vibrations of the system. In this paper, we build new Riemman invariants, one of which contains time-delay term, and utilize method of *characteristics* to rigorously prove the onset of chaos in the sense of *exponential growth of total variations* of the bounded gradient.

1. Introduction

During the past two decades, a number of papers have studied the chaotic vibration of the wave equation [2–11,13–19]. Significant progress has been made. Nevertheless, few papers have rigorously studied the dynamics of the wave equation system when the boundary conditions contain extra displacement feedback terms.

Recall the system considered in [7], as follows:

$$\begin{cases} w_{tt} - w_{xx} = 0, \quad x \in (0,1),\ t > 0, \\ w_x(0,t) = -\eta w_t(0,t), \quad \eta \neq 1, \\ w_x(1,t) = \alpha w_t(1,t) - \beta w_t^3(1,t) - \gamma w(1,t), \quad 0 < \alpha < 1,\ \beta > 0,\ t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 < x < 1. \end{cases} \tag{1.1}$$

When the parameter γ is sufficiently small, Chen et al. have rigorously analyzed the chaotic vibration of the system (1.1). Numerical experiments have shown that when $\gamma > 0$ is not small, chaotic vibration occurs

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http://dx.doi.org/10.1016/j.jmaa.2015.04.029

for some $\eta > 0$. However, there is no success yet for this case even now. The term $\gamma w(1,t)$ in the right boundary condition causes great technical difficulty in the analysis of its dynamics. A major reason is that the *method of characteristics* becomes inapplicable. For the moment, it is still an open problem.

In the standard PID (proportional, integral and differential) methodology of feedback control, the feedback of position or displacement is of utmost importance in problems such as tracking. Here the term γw corresponds precisely to the position or displacement term [7].

Therefore, it is of strong interest to consider the dynamics of wave equation with displacement involved in the boundary. In this paper, we study such the system as follows:

$$(\mathbf{S}) \begin{cases} w_{tt} - w_{xx} = 0, \quad x \in (0,1),\ t > 0, \\ w_x(0,t) = -\eta w_t(0,t) + \gamma w(0,t), \quad \eta \neq 1,\ \gamma \in \mathbb{R}, \\ w_x(1,t) = U(t), \quad t > 0, \\ w(x,0) = w_0(x) \in C^2([0,1]), \quad w_t(x,0) = w_1(x) \in C^1([0,1]), \end{cases} \tag{1.2}$$

where $U(t)$ corresponds to some kind of nonlinear input to be specified shortly. What is new here is the left-end boundary condition. In particular, when $\gamma > 0$, consider the energy function of the system and its derivative

$$E(t) = \frac{1}{2}[\int_0^1 w_t^2(x,t) + w_x^2(x,t)dx] + \frac{\gamma}{2}w^2(0,t),$$
$$E'(t) = \eta w_t^2(0,t) + w_t(1,t)U(t).$$

We see that if $\eta > 0$, energy is added to the system from the left boundary. So we may call it *energy injecting* boundary condition.

The new contribution of the paper is the finding of two new variables containing time-delayed displacement, i.e.

$$u(x,t) = \frac{w_x(x,t) + w_t(x,t)}{2},$$

and

$$v(x,t) = \begin{cases} \frac{w_x(x,t) - w_t(x,t)}{2} - \frac{\gamma}{1-\eta} w_0(0), \quad t \le x, \\ \frac{w_x(x,t) - w_t(x,t)}{2} - \frac{\gamma}{1-\eta} w(0,t-x), \quad t > x, \end{cases}$$

for $0 \le x \le 1$ and $t > 0$. Moreover, by designing appropriate boundary input $U(t)$ at the right side, we can rigorously study the dynamics of the system **(S)**, including chaotic vibration. More precisely,

$$U(t) = \begin{cases} \alpha w_t(1,t) - \beta[w_t(1,t) + \frac{\gamma}{1-\eta} w_0(0)]^3 + \frac{\gamma(1+\alpha)}{1-\eta} w_0(0), \quad t < 1, \\ \alpha w_t(1,t) - \beta[w_t(1,t) + \frac{\gamma}{1-\eta} w(0,t-1)]^3 + \frac{\gamma(1+\alpha)}{1-\eta} w(0,t-1), \quad t \ge 1. \end{cases} \tag{1.3}$$

Note that if $\gamma = 0$, it reduces to the classical van der Pol boundary condition.

This paper is organized as follows. In Section 2, we study the reflection relations and present compatibility conditions for solutions to be differentiable. Section 3 provides dynamic analysis of chaos in the sense of growth of total variations. Section 4 offers a concrete numerical example illustrating chaotic oscillations.