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The existence of solution for viscous Camassa–Holm equations on bounded domain in five dimensions

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ABSTRACT

The existence of global weak solution and local existence of strong solution for five-dimensional viscous Camassa–Holm equations on bounded domain are proved in this note. The global existence of strong solution is also proved when small initial data is given.

1. Background and the main result

Assume $L > 0$. The five-dimensional viscous Camassa–Holm (abbreviated as VCH) equations on $\mathcal{T} = [0, L]^5$ considered in this note are

$$\begin{cases} \frac{\partial(u-\alpha^2\Delta u)}{\partial t} - \nu\Delta(u-\alpha^2\Delta u) + (u\cdot\nabla)(u-\alpha^2\Delta u) + \sum_{j=1}^{m}(u-\alpha^2\Delta u)_j\nabla u_j + \nabla\frac{P}{\rho} = f, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \\ u \text{ is periodic on } \mathcal{T}, \end{cases} \tag{1.1}$$

where $u(x,t) = (u_1(x,t), \cdots, u_5(x,t))$ is the velocity of the fluid at point $x = (x_1, \cdots, x_5)$ at time t, $\frac{P}{\rho} = \frac{\pi}{\rho} + |u|^2 - \alpha^2(u\cdot\Delta u)$ is the modified pressure, while π is the pressure, $\nu > 0$ is the constant viscosity and ρ is a constant density, $\alpha > 0$ is scale parameter, at the limit $\alpha = 0$ one obtains the Navier–Stokes equations, the function f is a given body forcing.

The viscous Camassa–Holm equations first emerge in [7]. They average the motion of small scales of the Navier–Stokes equations. $\alpha > 0$ is a scale at which the fluid motion is averaged. Specifically, for any fixed α, VCH equations are able to capture accurately the motion of the fluid at scales larger than α while averaging

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out the motion of the fluid at scales smaller than α. The authors in [6] find the relationship between the Navier–Stokes equations (Eulerian formulation) and the VCH equations (Lagrangian formulation in some sense). Specifically, the unique weak solution of three-dimensional VCH equations converges to the weak solution of the three-dimensional Navier–Stokes equations when the scale converges to zero.

The global existence of solution $u \in L^2([0,T], H^3) \cap L^\infty([0,T]; H^2) \cap W^{1,\infty}([0,T]; L^2)$ has been proved for the equations on 2D periodic box in [10] with the assumption that $f \in L^2$, $u_0 \in H^2$. Assume $f \in L^2$, $u_0 \in H^1$, the unique existence of regular solution $u \in L^\infty_{loc}((0,T], H^3)$ for the equations on 3D periodic box has been proved in [6]. Decay of solution and the unique existence of global weak solution $u \in L^\infty([0,T], H^2) \bigcap L^2([0,T], H^3)$ for the equations on bounded domain or in the whole space R^m has been obtained in [3] for $m = 2, 3, 4$ under the conditions that $f = 0$ and $u_0 \in H^2$. Assume $f \in L^2([0,T]; H)$ $u_0 \in H^1$, the unique existence of weak solution for the equations on mD ($m = 2, 3, 4$) periodic box has been proved in [20]. Assume $f = 0$, $u_0 \in H^{2+s}(R^m)$ with $s > \frac{m}{2} - 1$, the regularity criteria for the equations has been considered in [21] for $m = 2, 3, 4$. Assume $f = 0$, $u_0 \in H^s$ with $s > \frac{m}{2} + 1$ and $m = 2, 3$, the uniqueness and smoothness of the global solution for VCH equations on m-dimensional Riemannian manifold with certain boundary conditions (Dirichlet, Neumann, and Mixed type boundary conditions) are proved in [15]. The higher dimensional cases are also considered in [15]. One can find in [2,8,9,16,18,19] for more research works about VCH equations. Note that VCH equations are formally similar to the Lagrangian averaged Navier–Stokes equations. The research history of Lagrangian averaged Navier–Stokes equations can be found in [1,5,11–14].

Above all, whether there exists a solution for the five-dimensional VCH equations is still an open problem. In this note the VCH equations on 5D periodic box are considered. The existence of global weak solution and unique existence of local strong solution for the equations are proved. The strong solution is proved globally when small initial data is given.

The techniques of the proof in [3,6,10,21] are energy estimate and Sobolev imbedding. While additional interpolations between different Sobolev imbedding are used in [20] and this note. The proof idea for $m = 2, 3, 4$ case and $m = 5$ case is almost similar. Because of the higher dimension, some imbedding and interpolations used for $m = 2, 3, 4$ case cannot be used for $m = 5$ case. Also because of this, the uniqueness of weak solution for $m = 5$ case cannot be proved in this paper, though it is true for $m = 2, 3, 4$ case [20].

One needs some notations to state the result.

Assume $\int_{\mathcal{T}} f dx = \int_{\mathcal{T}} u_0 dx = 0$ for simplicity. $\int_{\mathcal{T}} u dx = 0$ will be obtained after integration by parts from (1.1). Let $\mathcal{V} = \{\Phi \in C^\infty_{per}(\mathcal{T})^5; \nabla \cdot \Phi = 0, \int_{\mathcal{T}} \Phi dx = 0\}$, where $C^\infty_{per}(\mathcal{T})^5$ denotes the space of all $\mathcal{T}$-periodic, C^∞ vector fields defined on $\mathcal{T}$. H and V stand for the closure of $\mathcal{V}$ in $L^2(\mathcal{T})^5$ and $H^1(\mathcal{T})^5$ respectively. $(\cdot,\cdot)$ and $|\cdot|$ will be used to denote the scalar product and norm in H. The scalar product and norm in V is denoted by $((\cdot,\cdot))$ and $\|\cdot\|$. Let $A = P(-\Delta)$ is the abstract Stokes operator with domain $D(A) = H^2(\mathcal{T})^5 \bigcap V$, where P is the Leray projector. Under space periodic boundary conditions $A = -\Delta|_{D(A)}$ is a self-adjoint positive operator which is an isomorphism from V to V' (the dual of V). Hence A has eigenvalues $\{\lambda_j\}_{j=1}^\infty$ such that $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to +\infty$. Moreover $V = D(A^{\frac{1}{2}})$, $||\cdot|| = |A^{\frac{1}{2}} \cdot|$, $((\cdot,\cdot)) = (A^{\frac{1}{2}}\cdot, A^{\frac{1}{2}}\cdot)$. By virtue of Poincaré inequality one can show that there is a constant c such that

$$c^{-1}|Aw| \le ||w||_{H^2} \le c|Aw|, \qquad \forall\, w \in D(A)$$

and

$$c^{-1}|A^{\frac{1}{2}}w| \le ||w||_{H^1} \le c|A^{\frac{1}{2}}w|, \qquad \forall\, w \in V.$$

Setting

$$B(\phi,\psi) = P\Big((\phi \cdot \nabla)\psi\Big), \quad B(\psi)\phi = B(\phi,\psi), \qquad \forall\, \phi,\psi \in V,$$

where $B(\psi)$ is a linear operator acting on ϕ for every fixed ψ.