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Existence of multiple positive weak solutions and estimates for extremal values to a class of elliptic problems with Hardy term and singular nonlinearity

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ABSTRACT

This paper is devoted to the study of a class of elliptic equations of the form:

$$-\Delta u - \frac{\lambda}{|x|^2}u = h(x)u^{-q} + \mu W(x)u^p, \quad x \in \Omega\backslash\{0\}$$

with Dirichlet boundary conditions, where $0 \in \Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $\mu > 0$ is a parameter, $0 < \lambda < \Lambda = \frac{(N-2)^2}{4}$, $0 < q < 1 < p < 2^* - 1 = \frac{N+2}{N-2}$, $h(x) > 0$ and $W(x)$ is a given function with the set $\{x \in \Omega : W(x) > 0\}$ of positive measure. Using variational methods, we establish some existence and multiplicity of positive solutions and provide uniform estimates of extremal values for the problem.

1. Introduction

In this paper, we consider a class of elliptic problems with Hardy term and singular nonlinearity of the following form:

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2}u = h(x)u^{-q} + \mu W(x)u^p & \text{in } \Omega\backslash\{0\}, \\ u(x) > 0 & \text{in } \Omega\backslash\{0\}, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Δ is the Laplace operator and $\mu > 0$ is a parameter. $0 \in \Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, $0 < \lambda < \Lambda = \frac{(N-2)^2}{4}$ and $0 < q < 1 < p < 2^* - 1$, where $2^* = \frac{2N}{N-2}$ is the so-called critical Sobolev exponent. $h(x) > 0$ and $W(x)$ is a given function with the set $\{x \in \Omega : W(x) > 0\}$ of positive measure (that is, $W \geq 0$ or W changes sign). We will assume throughout the paper that $h, W \in C(\bar{\Omega})$.

The constant $\Lambda = \frac{(N-2)^2}{4}$ is the best constant in the Hardy inequality (see [13, p. 1879, Theorem 1.2] and apply it with $\gamma = 0$):

$$\int_\Omega \frac{u^2}{|x|^2} dx \leq \frac{1}{\Lambda} \int_\Omega |\nabla u|^2 dx \quad \text{for all } u \in H_0^1(\Omega)$$

and it is not achieved. Consequently, when $\lambda < \Lambda$, the operator $-\Delta - \frac{\lambda}{|x|^2}$ is coercive in $H_0^1(\Omega)$. This turns out to be crucial, since [4, p. 124, Theorem 2.2] implies that if $\lambda > \Lambda$, there is no nonnegative u, $u \not\equiv 0$ such that

$$-\Delta u - \frac{\lambda}{|x|^2} u \geq 0$$

and hence no solution of problem (1.1), even in the weak sense.

The general problem

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u = h(x) u^{-q} + \mu W(x) u^p & \text{in } \Omega \backslash \{0\}, \\ u(x) > 0 & \text{in } \Omega \backslash \{0\}, \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \tag{$P_{\lambda,\mu}$}$$

has been studied extensively. When $\lambda = 0$, $q > 0$, $p \geq 1$ and $h, W \equiv 1$, M.M. Coclite and G. Palamieri [11, p. 1318, Corollary 4] showed that there exists $\mu_1 > 0$ such that problem $(P_{\lambda,\mu})$ has at least one solution for every $\mu < \mu_1$ and has no solution for any $\mu > \mu_1$. Under different assumptions on p, q, h, W, J. Hernández, F.J. Mancebo and J.M. Vega [16] obtained similar conclusions for problem $(P_{0,\mu})$ (one can see pp. 56–59, Theorem 3.21, Remark 3.22, Theorem 3.25(ii) and Theorem 3.29, and apply them with $\mathcal{L} = -\Delta$). Furthermore, the multiplicity of positive solutions was claimed for problem $(P_{0,\mu})$ with various p, q, h, W, such as Y. Sun, S. Wu and Y. Long [22, p. 518, Theorem 1], Y. Sun and S. Li [20, p. 2637, Theorem 1] and Y. Sun and S. Wu [21, p. 1263, Theorem 1; p. 1276, Theorem 2]. We also refer the interested readers to [10,12,15–18] and the references therein for various results of solutions of problem $(P_{\lambda,\mu})$ in the case of $\lambda = 0$. When $0 < \lambda < \Lambda = \frac{(N-2)^2}{4}$, $0 < q < 1 < p = 2^* - 1$ and $h, W \equiv 1$, by a scaling (see Appendix A(1)), J. Chen and E.M. Rocha [9, p. 414, Theorem 1.1] proved that there exists $\mu_2 > 0$ such that problem $(P_{\lambda,\mu})$ has at least two positive solutions for any $\mu \in (0, \mu_2)$.

Observing these results above, we do not see any multiplicity of positive solutions about problem $(P_{\lambda,\mu})$ in the case of the Hardy term, singular nonlinearity u^{-q} and the weight functions h, W with W sign-changing, so it is natural to ask what the case would be. Our goal of this paper is to show how variational methods can be used to establish some existence and multiplicity results for problem (1.1) when $\mu \in (0, T_\lambda)$ for some $T_\lambda > 0$ (see Theorem 1.1) and obtain uniform estimates for extremal values $\mu^* = \mu^*(N, \Omega, \lambda, q, p) > 0$ for problem (1.1) provided $h, W \equiv 1$ (see Lemma 5.1 and Theorem 1.3).

On $H_0^1(\Omega)$, we use the norm

$$\|u\|_\lambda^2 = \int_\Omega \left(|\nabla u|^2 - \frac{\lambda}{|x|^2} u^2 \right) dx.$$