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Even convexity, subdifferentiability, and Γ-regularization in general topological vector spaces

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ABSTRACT

In this paper we provide new results on even convexity and extend some others to the framework of general topological vector spaces. We first present a characterization of the even convexity of an extended real-valued function at a point. We then establish the links between even convexity and subdifferentiability and the Γ-regularization of a given function. Consequently, we derive a sufficient condition for strong duality fulfillment in convex optimization problems.

1. Introduction

The notion of even convexity appeared for the first time in the fifties when Fenchel [5] introduced the *evenly convex sets* as those which are intersections of open halfspaces. Over the years since then, the evenly convex sets have made occasional appearances in the literature (see, for instance, [2,6–8,10]). Recently, even convexity emerges again in [16] where the *evenly convex functions* are defined as those whose epigraphs are evenly convex sets. Previously, in the eighties, evenly quasiconvex functions (those with evenly convex sublevel sets) were introduced in quasiconvex programming [9,14]. Although the definitions of evenly convex set and evenly convex function were given in a finite-dimensional space, they have been also considered in any separated locally convex space [4,11,19] in a natural way.

On the other hand, it is well-known that the classical subdifferential for convex functions and the Γ-regularization of Moreau [12] play a significant role in convex optimization [1,15]. However, no systematic relationship has been established between these notions and even convexity. This fact has motivated us to study the links between these three main concepts in optimization. For that purpose, we first extend

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some given results on even convexity to general topological vector spaces and we get some new results with applications in convex conjugacy and duality.

The organization of the paper is as follows. In Section 2 we first introduce the necessary tools from convex analysis (see, for instance, [1,3,15,17,20]), and we study the structure of the evenly convex hull of the so-called ascending subsets of $X \times \mathbb{R}$. As a consequence, we obtain a geometrical characterization of the even convexity of an extended real-valued function at a given point. We establish in Section 3 a characterization of the subdifferentiability of a function at a given point in terms of the even convexity of the strict epigraph of the function. We provide in Section 4 a formula for the evenly convex hull of a function in terms of its Γ-regularized function and the valley function of its effective domain. Thus, we recover some results established in the frame of locally convex topological vector spaces, without assuming the properness of the function. Section 5 is devoted to evenly convex conjugacy in general topological vector spaces. Finally, Section 6 shows an application of our results to convex optimization duality.

2. Even convexity in the product space $X \times \mathbb{R}$

We begin this section by fixing notation and preliminaries. Unless otherwise specified, throughout the paper X will denote a separated and real topological vector space. We denote by X^* the topological dual space of X, and set $\langle x^*, x\rangle := x^*(x)$ for $(x, x^*) \in X \times X^*$. The corresponding topological dual space of $X \times \mathbb{R}$ is identified with $X^* \times \mathbb{R}$ by means of the bilinear form

$$\langle (x^*, s), (x, r)\rangle := \langle x^*, x\rangle + sr, \quad (x^*, s) \in X^* \times \mathbb{R}, (x, r) \in X \times \mathbb{R}.$$

For an extended real-valued function $h : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we denote by $\operatorname{epi} h := \{(x, r) \in X \times \mathbb{R} : h(x) \le r\}$ its epigraph, by $\operatorname{epi}_s h := \{(x, r) \in X \times \mathbb{R} : h(x) < r\}$ its strict epigraph, and by $\operatorname{dom} h := \{x \in X : h(x) < +\infty\}$ its effective domain. The function h is convex provided that $\operatorname{epi} h$ is convex or, equivalently, if $\operatorname{epi}_s h$ is convex. One says that h is lower semicontinuous (lsc, in brief) at a point $\bar{x} \in X$ if, for any real number $t < h(\bar{x})$, there exists a neighborhood V of $\bar{x}$ such that $t < h(x)$ for any $x \in V$. Moreover, h is said to be lsc on $A \subset X$ if it is lsc at each point of A. Thus, h is lsc on X provided that $\operatorname{epi} h$ is a closed subset of $X \times \mathbb{R}$ or, equivalently, if the sublevel sets $[h \le r] := \{x \in X : h(x) \le r\}$, $r \in \mathbb{R}$, are all closed. We denote by $\bar{h}$ the lsc hull of h. It holds that $\operatorname{epi} \bar{h} = \overline{\operatorname{epi} h}$, the closure of $\operatorname{epi} h$ in $X \times \mathbb{R}$.

Given $A \subset X$, we shall denote by $\operatorname{co} A$ (respectively, $\overline{\operatorname{co}}\, A$) the convex (respectively, the closed convex) hull of A. We also associate to the set A its *indicator function* i_A defined on X by $i_A(x) := 0$ if $x \in A$, $i_A(x) := +\infty$ if $x \notin A$, and its *valley function* v_A defined on X by $v_A(x) := -\infty$ if $x \in A$, $v_A(x) := +\infty$ if $x \notin A$. The *recession cone* of a nonempty convex set $C \subset X$ is defined as

$$O^+(C) := \{d \in X : c + \lambda d \in C, \forall c \in C, \forall \lambda \ge 0\}.$$

Recall that a subset of X is said to be *evenly convex* [5] if it is an arbitrary intersection of open halfspaces of X. Hence, given $A \subset X$, there exists the smallest evenly convex set containing A, and it is denoted by $\operatorname{eco} A$. For any $\bar{x} \in X$, it holds

$$\bar{x} \notin \operatorname{eco} A \;\Leftrightarrow\; \exists\, x^* \in X^* : \langle x^*, \bar{x}\rangle > \langle x^*, x\rangle\,,\ \forall x \in A. \tag{1}$$

A function $h : X \to \overline{\mathbb{R}}$ is said to be *evenly convex* [16] if its epigraph $\operatorname{epi} h$ is an evenly convex set in $X \times \mathbb{R}$. Since the intersection of infinitely many evenly convex sets is evenly convex, the supremum of evenly convex functions is again an evenly convex function, and so, any function $h : X \to \overline{\mathbb{R}}$ admits a greatest evenly convex minorant denoted by $\operatorname{eco} h$. Throughout the paper, we adopt the rule $(+\infty) + (-\infty) = +\infty$, and use the corresponding properties (see [10,13]).