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Exponential stability for second order evolutionary problems

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A B S T R A C T

We study the exponential stability of evolutionary equations. The focus is laid on second order problems and we provide a way to rewrite them as a suitable first order evolutionary equation, for which the stability can be proved by using frequency domain methods. The problem class under consideration is broad enough to cover integro-differential equations, delay-equations and classical evolution equations within a unified framework.

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1. Introduction

Evolutionary equations, as they were first introduced by Picard [12,13,15], consist of a first order differential equation on $\mathbb{R}$ as the time-line

$$\partial_0 v + Au = f,$$

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where ∂_0 denotes the derivative with respect to time and A is a suitable closed linear operator on a Hilbert space (frequently a block-operator-matrix with spatial differential operators as entries). The function u and v are the unknowns, while f is a given source term. This simple equation is completed by a so-called linear material law linking u and v:

$$v = \mathcal{M}u,$$

where $\mathcal{M}$ is an operator acting in time and space. Thus, the differential equation becomes

$$\left(\partial_0\mathcal{M} + A\right) u = f \tag{1}$$

a so-called evolutionary problem. The operator sum on the left-hand side will be established in a suitable Hilbert space and thus, the well-posedness of (1) relies on the bounded invertibility of that operator sum. For doing so, one establishes the time-derivative ∂_0 as a normal boundedly invertible operator in a suitable exponentially weighted L_2-space. With the spectral representation of this normal operator at hand, one specifies the operator $\mathcal{M}$ as to be an analytic operator-valued function of ∂_0^{-1}. Then $\mathcal{M}$ enjoys the property that it is causal due to the Theorem of Paley and Wiener (see e.g. [19, Theorem 19.2]). Although the requirement of analyticity seems to be very strong, these operators cover a broad class of possible space–time operators like convolutions with suitable kernels naturally arising in the study of integro-differential equations (see e.g. [22]), translations with respect to time as they occur in delay equations (see [10]) as well as fractional derivatives (see [14]). Thus, the setting of evolutionary equations provides a unified framework for a broad class of partial differential equations. We note that the causality of $\mathcal{M}$ also yields the causality of the solution operator $(\partial_0\mathcal{M} + A)^{-1}$ of our evolutionary problem (1), which can be seen as a characterizing property of evolutionary processes.

After establishing the well-posedness of (1), one is interested in qualitative properties of the solution u. A first property, which can be discussed, is the asymptotic behavior of u, especially the question of exponential stability. The study of stability for differential equations goes back to Lyapunov and a lot of approaches has been developed to tackle this question over the last decades. We just like to mention some classical results for evolution equations, using the framework of strongly continuous semigroups, like Datko's Lemma [6] or the Theorem of Gearhart–Prüss [9,17] (see also [7, Chapter V] for the asymptotics of semigroups). Unfortunately, these results are not applicable to evolutionary equations. The main reason for that is that the solution u of (1) is not continuous, unless the right-hand side f is regular, so that point-wise estimates for the solution u (and this is how exponential stability is usually defined) do not make any sense. Hence, we need to introduce a more general notion of exponential stability for that class of problems. This was done by the author in [20,21] (see also Subsection 2.2 in this article), where also sufficient conditions on the material law $\mathcal{M}$ to obtain exponential stability were derived.

The main purpose of this article is to study the exponential stability of evolutionary problems of second order in time and space, i.e. to equations of the form

$$\left(\partial_0^2\mathcal{M} + C^*C\right) u = f, \tag{2}$$

where C is a densely defined closed linear operator, which is assumed to be boundedly invertible. For doing so, we need to rewrite the above problem as an evolutionary equation of first order in time. As it turns out there are several ways to do this yielding a family of new material law operators $(\mathcal{M}_d)_{d>0}$, such that (2) can be written as

$$\left(\partial_0\mathcal{M}_d + \begin{pmatrix} 0 & C^* \\ -C & 0 \end{pmatrix}\right) \begin{pmatrix} \partial_0 u + du \\ Cu \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \tag{3}$$