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Fractal measures with uniform marginals

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A B S T R A C T

We provide several constructions of self-affine probability measures on the unit square with uniform marginals. These constructions include and extend constructions of previous authors and are parameterized in a natural way. In addition, for each different construction we determine the dimension of the parameter space and thus the level of flexibility (for instance, for approximation purposes) each construction allows. Finally, we give some simple approximation results showing how to approximate any measure with uniform marginals on the unit square with a fractal measure resulting from one of our constructions.

1. Introduction

Bivariate distributions can be described using the two marginal distributions along with a *copula*. The copula encodes the dependence structure of the two components by using the cumulative distribution function of a distribution with uniform marginals. This disentangles the marginal distributions from the "dependence structure."

Copulas have recently become quite important in modeling financial risk where the dependence between several factors is of crucial importance. In this area, it is necessary to fit a multivariate dependence model to data given some idea of the marginals and some (generally weak) information about their dependence. To do this one needs models of copulas, most often parametric families. A relatively new alternative is to consider copulas with a fractal structure [3–6]. In these papers, the authors construct copulas with fractal support and investigate some of their properties (both geometric and probabilistic).

Instead of using a copula, the dependence structure can also be encoded by using a probability distribution (measure) with uniform marginals and we take this approach. Our first construction, using "product partitions," of such measures was also implicitly used by the previous authors (in the context of constructing the associated copula). We repeat this construction to establish notation and as a starting point for the

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other more general constructions. We also provide a count of the number of free parameters (dimension of the solution space) for all the constructions, which was not previously done. The next construction, in Section 3.2, is a new construction and is based on a partition of $[0,1]^2$ into blocks that is not a "product partition." The necessary conditions to obtain uniform marginals are more complicated in this situation and analyzing the number of free parameters is also more difficult. In Section 4, we provide a "Markovian" construction (also new). This construction is related to so-called "graph-directed" (see [8]) or "recurrent" (see [1]) IFS and results in "locally" self-affine measures with uniform marginals. Finally, we close with some comments on some further directions for generalizations and possible avenues for future exploration.

2. Mathematical preliminaries

We first need to remind the reader of some of the basics in the theory of iterated function systems with probabilities. For a more complete introduction, see [7, Chapter 2]. An iterated function system with probabilities (IFSP) on a complete metric space $\mathbb{X}$ is a collection of self-maps $w_i : \mathbb{X} \to \mathbb{X}$, $i = 1, 2, \ldots, N$, along with associated probabilities p_i. Corresponding to an IFSP is a Markov operator $\mathbb{M}$ acting on the set of all Borel probability measures $\mu \in \mathcal{P}(\mathbb{X})$ and defined by

$$\mathbb{M}\mu(B) = \sum_i p_i \mu(w_i^{-1}(B)), \tag{1}$$

for an arbitrary Borel set B. Using the notation $\mu \llcorner A$ for the restriction measure defined by $(\mu \llcorner A)(B) = \mu(A \cap B)$, we see that $\mathbb{M}\mu \llcorner w_i(\mathbb{X})$ is a "distorted" (by w_i) and scaled (by p_i) version of μ. Thus any fixed point of $\mathbb{M}$ as defined by (1) will be "fractal" in the sense of consisting of a combination of "smaller" copies of itself.

The convergence of the iterates of $\mathbb{M}$ is usually analyzed with the help of the Monge–Kantorovich metric on $\mathcal{P}(\mathbb{X})$:

$$d_{MK}(\mu, \nu) = \sup \left\{ \int_{\mathbb{X}} f \; d(\mu - \nu) : f : \mathbb{X} \to \mathbb{R}, |f(x) - f(y)| \leq d(x, y) \right\}. \tag{2}$$

In general, d_{MK} might be infinite (e.g., $\mathbb{X} = \mathbb{R}$, μ is a point-mass at 0 and ν is a Cauchy random variable) so it is necessary to place restrictions on the collection of measures. For our purposes, it is enough to consider compact spaces $\mathbb{X}$, in which case $d_{MK}(\mu, \nu) \leq \operatorname{diam}(\mathbb{X})$ and the space $\mathcal{P}(\mathbb{X})$ is also compact.

If w_i has Lipschitz constant c_i, then $\mathbb{M}$ is Lipschitz (in the Monge–Kantorovich metric) with factor no greater than $\sum_i p_i c_i$. When $\sum_i p_i c_i < 1$, we say that the IFSP is *average contractive* and, by the contraction mapping theorem, $\mathbb{M}$ has a unique fixed point $\hat{\mu}$, the *invariant distribution*.

Throughout the rest of the paper we take $\mathbb{X} = [0,1]^2$, and so $\mathbb{X}$ is compact. The extension to $[0,1]^d$ for $d > 2$ is straightforward.

3. Basic construction

We start with a collection of closed rectangular blocks $B_i \subset [0,1]^2$ such that $\cup_i B_i = [0,1]^2$ and $int(B_i) \cap int(B_j) = \emptyset$ for $i \neq j$. For each i, the map $w_i : [0,1]^2 \to B_i$ is set to be affine with $w_i(0,0)$ being the lower left-hand corner of B_i and $w_i(1,1)$ being the upper right-hand corner of B_i. The aim is to choose probabilities p_i so that the invariant measure of $\{w_i, p_i\}$ has uniform marginals. We wish to decide when this is possible and determine the dimensionality of the collection of p_i for which it is possible.

Let $\mathcal{P}_u \subset \mathcal{P}([0,1]^2)$ be those Borel probability measures with uniform marginals. The idea behind our construction (and the constructions of previous authors) is to choose the probabilities, p_i, in such a way that