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Closure of Hardy spaces in the Bloch space ☆

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ABSTRACT

A description of the Bloch functions that can be approximated in the Bloch norm by functions in the Hardy space H^p of the unit ball of $\mathbb{C}^n$ for $0 < p < \infty$ is given. When $0 < p \le 1$, the result is new even in the case of the unit disk.

1. Introduction

Let $\mathbb{D}$ and $\mathbb{T}$ be, respectively, the unit disk and the unit circle of the complex plane $\mathbb{C}$. For $0 < p < \infty$, recall that the Hardy space $H^p(\mathbb{D})$ is the space of analytic functions f in the unit disk such that

$$\|f\|_p^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} < +\infty.$$

For $p = \infty$, $H^\infty(\mathbb{D})$ is the space of all bounded analytic functions in the unit disk. Recall also that the Bloch space $\mathcal{B}(\mathbb{D})$ is formed by the analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} (1 - |z|^2)\, |f'(z)| < \infty.$$

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In [10], a characterization of the closure in the Bloch norm of $H^p \cap \mathcal{B}$ for $1 < p < \infty$ was given in terms of the area of certain non-tangential level sets of the Bloch function: given a function $f \in \mathcal{B}$ and $\varepsilon > 0$ define the level set of f as

$$\Omega_\varepsilon(f) := \{z \in \mathbb{D} \ : \ (1 - |z|^2)|f'(z)| \geq \varepsilon\}.$$

Recall that a Stolz angle with vertex in $\zeta \in \mathbb{T}$ is the set

$$\Gamma(\zeta) = \Gamma_\alpha(\zeta) := \{z \in \mathbb{D} : |z - \zeta| < \frac{\alpha}{2}(1 - |z|)\},$$

with $\alpha > 2$, and that

$$A_h(\Omega) := \int_\Omega \frac{dA(z)}{(1 - |z|^2)^2},$$

where $dA(z)$ is the area measure in $\mathbb{D}$, represents the hyperbolic area of $\Omega \subset \mathbb{D}$. Then the result is the following:

Theorem A. *Let f be a function in the Bloch space $\mathcal{B}$ and $1 < p < \infty$. Then f is in the closure in the Bloch norm of $\mathcal{B} \cap H^p$ if and only if for any $\varepsilon > 0$ the function $A_h(\Gamma(\zeta) \cap \Omega_\varepsilon(f))^{1/2}$ is in $L^p(\mathbb{T})$.*

A basic tool in the proof of this result was the characterization of Hardy spaces in terms of the area function (a result due to J. Marcinkiewicz and A. Zygmund [9] for $p > 1$, and extended to the case $0 < p \leq 1$ by A. Calderón [3]), that is, for $0 < p < \infty$, a function f is in H^p if and only if its corresponding Lusin Area function

$$A(f)(\zeta) = \left(\int_{\Gamma(\zeta)} |f'(z)|^2 dA(z) \right)^{1/2}$$

is in $L^p(\mathbb{T})$. The proof of Theorem A was based on a previous result by P. Jones on the closure of *BMOA* in $\mathcal{B}$ (see [6]). The duality argument given in the proof in [10] cannot be used for $0 < p \leq 1$, so that this case requires of new techniques. In this paper we solve the case $0 < p \leq 1$. It turns out that the proof given works equally for all $0 < p < \infty$, and furthermore, it may be done in the open unit ball $\mathbb{B}_n$ of the n-dimensional complex space $\mathbb{C}^n$. The case $p = \infty$ is still an open problem, and will be discussed in the last section.

Now we are going to introduce some notation. For $z, w \in \mathbb{C}^n$, let

$$\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n.$$

Hence, $|z|^2 = \langle z, z \rangle$. In this context, for $0 < p < \infty$ the Hardy space $H^p(\mathbb{B}_n)$ consists of those holomorphic functions f on $\mathbb{B}_n$ such that

$$\|f\|_p^p = \sup_{0<r<1} \int_{\mathbb{S}_n} |f(r\zeta)|^p \, d\sigma(\zeta) < +\infty,$$

where $\mathbb{S}_n$ denotes the unit sphere in $\mathbb{C}^n$ and σ is the normalized surface measure on $\mathbb{S}_n$. As in the case for $n = 1$, for $p = \infty$ the corresponding space $H^\infty(\mathbb{B}_n)$ is the space of bounded holomorphic functions defined on $\mathbb{B}_n$.