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Global bounded solutions and their asymptotic properties under small initial data condition in a two-dimensional chemotaxis system for two species

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ABSTRACT

In this paper, a fully parabolic chemotaxis system for two species

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \nabla \cdot (v \nabla w), & x \in \Omega,\ t > 0, \\ w_t = \Delta w + u - w - vw, & x \in \Omega,\ t > 0 \end{cases}$$

is considered under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^2$. We obtain the global boundedness and asymptotic behavior with small initial data condition in critical space. More precisely, it is proved that one can find a small $\varepsilon_0 > 0$ such that for any initial data (u_0, v_0, w_0) satisfying $||u_0||_{L^1(\Omega)} < \varepsilon_0$ and $||\nabla w_0||_{L^2(\Omega)} < \varepsilon_0$, the solution of the problem above is global in time and bounded. In addition, (u, v, w) converges to the steady state $(m_1, m_2, \frac{m_1}{1+m_2})$ as $t \to \infty$, where $m_1 := \frac{1}{|\Omega|} \int_\Omega u_0$ and $m_2 := \frac{1}{|\Omega|} \int_\Omega v_0$.

1. Introduction

We consider the following chemotaxis system for two species, both of which are attracted by the same chemical stimulus,

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$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \nabla \cdot (v \nabla w), & x \in \Omega,\ t > 0, \\ w_t = \Delta w + u - w - vw, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in \Omega \end{cases} \tag{1.1}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^2$, where $u(x,t)$ and $v(x,t)$ denote the population densities, respectively, and $w(x,t)$ represents the concentration of the chemoattractant.

Processes, which describe the migration of cells in response to a chemical signal produced by the cells, also referred to as chemotaxis, play an important role in the interaction of cells. A classical model, as introduced by Keller and Segel [11], is given by

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ w_t = \Delta w - w + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

which has been extensively studied for the past decades, see for example [10,19] and the review papers [7,8,6] and the references therein. Among the large quantity of related researches, to decide whether solutions are global in time or blow up in finite time seems to be quite challenging. There are many results concerning global bounded solutions and unbounded solutions of (1.2). For instance, it is known that the solutions of Neumann initial–boundary value problem in bounded domains $\Omega \subset \mathbb{R}^n$ associated with (1.2) exist globally in time and are bounded whenever $n = 1$, or in two dimensions with $\int_\Omega u_0 < 8\pi$ in the radial and $\int_\Omega u_0 < 4\pi$ in the nonradial case. However, for any $\int_\Omega u_0 > 4\pi$ satisfying $\int_\Omega u_0 \bar{\in} \{4k\pi | k \in \mathbb{N}\}$, there exists initial data (u_0, w_0) such that the corresponding solution of (1.2) blows up in either finite time or infinite time. In higher dimensions, Winkler [19] proved that for $q > \frac{n}{2}$, $p > n$ and suitable small $\varepsilon > 0$ satisfying $||u_0||_{L^q(\Omega)} < \varepsilon$ and $||\nabla w_0||_{L^p(\Omega)} < \varepsilon$, the solution of system (1.2) exists globally and is bounded. Moreover, such solutions converge to constant equilibria in the large time limit. In a recent work [1], the author extended this result in a corresponding critical case, that is for $q = \frac{n}{2}$ and $p = n$. For finite-time blow-up result of (1.2) we refer to Winkler [21].

In corresponding variants of (1.2) with signal absorption is approached by

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ w_t = \Delta w - uw, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

where the signal is consumed by the cells, rather than produced by the cells. The quantity w is bounded by its initial norm in $L^\infty(\Omega)$. Tao [16] showed that in the dimension $n \geq 2$, the classical solution of system (1.3) exists globally in time and is uniformly bounded under the condition $0 < \chi \leq \frac{1}{6(n+1)||w_0||_{L^\infty(\Omega)}}$; In three dimensions, Tao and Winkler [17] proved that (1.3) possesses a global weak solution for which there exists $T > 0$ such that (u, w) is bounded and smooth in $\Omega \times (T, \infty)$ and such solutions approach the steady state in the large time limit.

Recently, the chemotaxis systems with several species and stimulus were investigated by some authors, see for instance [5,23] and the references therein. We also refer to Horstmann [9] for a broad overview. System (1.4) in which two populations react on a single chemoattractant have been described, e.g., in Kelly et al. [12], Lauffenburger et al. [13], Lauffenburger and Calcagno [14] from an experimental point of view and studied by some authors from a mathematical view in Espejo et al. [4], for instance. They considered the following system of two-species drift–diffusion equations

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w), & x \in \Omega,\ t > 0, \\ \tau w_t = \Delta w - \gamma w + \alpha_1 u + \alpha_2 v, & x \in \Omega,\ t > 0. \end{cases} \tag{1.4}$$