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Noncommutative Hardy space associated with semi-finite subdiagonal algebras

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ABSTRACT

Let $\mathcal{M}$ be a von Neumann algebra with a faithful normal semi-finite trace τ, and let $\mathcal{A}$ be maximal subdiagonal algebra of $\mathcal{M}$. Then for $0 < p < \infty$, we define the noncommutative H^p-space. We obtain that the conjugation and Herglotz maps are bounded linear maps from $L^p(\mathcal{M})$ into $L^p(\mathcal{M})$ for $1 < p < \infty$, and continuous map from $L^1(\mathcal{M})$ into $L^{1,\infty}(\mathcal{M})$. We also give the dual space of $H^p(\mathcal{A})$ for $1 \leq p < \infty$, and extend Pisier's interpolation theorem to this case.

1. Introduction

In [1], Arveson introduced the notion of finite, maximal, subdiagonal algebras $\mathcal{A}$ of $\mathcal{M}$, as noncommutative analogues of weak*-Dirichlet algebras, for von Neumann algebra $\mathcal{M}$ with a faithful normal finite trace. Subsequently several authors studied the (noncommutative) $\mathcal{H}^p$-spaces associated with such algebras [5,15,16,18,21–23]. Arveson [1] proved a Szegö type factorization theorem, of which some extensions can be found in [23,13,20]. In [16] and [21] the authors studied the closure of $\mathcal{A}$ in the noncommutative L^p-space $L^p(\mathcal{M})$, as an analogue of the classical Hardy space $H^p(\mathbb{T})$. They obtained generalizations of several classical results, including a Riesz factorization theorem for $H^1(\mathcal{A})$, a Riesz–Bochner theorem on the existence and boundedness of harmonic conjugates, a projection from $L^p(\mathcal{M})$ to $H^p(\mathcal{A})$, the duality between $H^p(\mathcal{A})$ and $H^q(\mathcal{A})$, and continuity of the Hilbert transform from $L^1(\mathcal{M})$ into $L^{1,\infty}(\mathcal{M})$. In [17] the authors identified the dual of $H^1(\mathcal{A})$ with a noncommutative analogue of the *BMO* space as in Fefferman's classical result.

The main objective of this paper is to obtain a generalization of the results in [16,17,19,21], for the semi-finite case.

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2. Preliminaries

Throughout this paper, we denote by $\mathcal{M}$ a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$ with a normal faithful semifinite trace τ. A closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $u^*xu = x$ for all unitary operators u which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If x is affiliated with $\mathcal{M}$, then x is said to be τ-measurable if for every $\varepsilon > 0$ there exists a projection $e \in \mathcal{M}$ such that $e(\mathcal{H}) \sqsubseteq D(x)$ and $\tau(e^\perp) < \varepsilon$ (where for any projection e we let $e^\perp = 1 - e$). The set of all τ-measurable operators will be denoted by $\overline{\mathcal{M}}$. The set $\overline{\mathcal{M}}$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product. For a positive self-adjoint operator $x = \int_0^\infty \lambda de_\lambda$ (the spectral decomposition) affiliated with $\mathcal{M}$, we set

$$\tau(x) = \sup_n \tau(\int_0^n \lambda de_\lambda) = \int_0^\infty \lambda \tau(e_\lambda).$$

For $0 < p < \infty$, $L^p(\mathcal{M})$ is defined as the set of all τ-measurable operators x affiliated with $\mathcal{M}$ such that

$$\|x\|_p = \tau(|x|^p)^{\frac{1}{p}} < \infty.$$

In addition, we put $L^\infty(\mathcal{M}) = \mathcal{M}$ and denote by $\|\cdot\|_\infty$ $(= \|\cdot\|)$ the usual operator norm. It is well known that $L^p(\mathcal{M})$ is a Banach space under $\|.\|_p$ $(1 \le p \le \infty)$ satisfying all the expected properties such as duality.

For a τ-measurable operator x, we define the distribution function of x by

$$\lambda_t(x) = \tau(e_{(t,\infty)}(|x|)), \quad t \ge 0,$$

where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ corresponding to the interval (t, ∞).

The following elementary properties of the distribution function $\lambda(x)$ will be used.

Lemma 2.1. *Let $x, y \in \overline{\mathcal{M}}$.*

(i) *If $0 \le x \le y$, then $\lambda(x) \le \lambda(y)$.*
(ii) *If $a \in \mathcal{M}$ is a contraction and $x \ge 0$, then $\lambda(a^*xa) \le \lambda(x)$.*
(iii) *If $t, s \in (0, \infty)$, then $\lambda_{s+t}(x + y) \le \lambda_s(x) + \lambda_t(y)$.*

Proof. (i) and (ii) follow from (i) and (ii) of Lemma 3 in [6].

(iii) It is known, but for the convenience of the reader, we add a short proof. By Lemma 4.3 of [11], there exist partial isometries u, v in $\mathcal{M}$ such that

$$|a + b| \le u|a|u^* + v|b|v^*.$$

Using (i) we obtain that

$$\tau(e_{(t+s,\infty)}(|a + b|)) \le \tau(e_{(t+s,\infty)}(u|a|u^* + v|b|v^*))$$

for all $t, s \in (0, \infty)$. Let $\xi \in e_{(t+s,\infty)}(u|a|u^* + v|b|v^*)(\mathcal{H})$, $\|\xi\| = 1$. Then $\|(u|a|u^* + v|b|v^*)\xi\| > s + t$. It follows that $\|u|a|u^*\xi\| > s$ or $\|v|b|v^*\xi\| > t$, so that

$$\xi \notin e_{[0,t]}(u|a|u^*)(\mathcal{H}) \cap e_{[0,s]}(v|b|v^*)(\mathcal{H}).$$