J. Math. Anal. Appl. 429 (2015) 1370–1373

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Note

A counterexample to a Frederico–Torres fractional Noether-type theorem

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A R T I C L E I N F O

Article history:
Received 20 February 2015
Available online 21 April 2015
Submitted by H. Frankowska

Keywords:
Noether's theorem
Fractional derivatives

A B S T R A C T

In this note we present a counterexample to Theorem 24 in G.S.F. Frederico and D.F.M. Torres (2010) [3].

1. Introduction

In 2010 Frederico and Torres presented, in [3], a Noether-type theorem (with transformation of time) for a variational problem depending on fractional derivatives. It reads as follows.

Theorem 1.1. *(See [3, Theorem 24].) If the functional*

$$I(q) = \int_a^b L\left(t, q(t), {}_a^{RC}D_b^\alpha q(t)\right) dt, \tag{1}$$

where $0 < \alpha < 1$ and L is a C^2 function of its arguments, is invariant (cf. Definition 1.4 below), then

$$\mathcal{D}_t^\alpha\left[\partial_3 L\left(t, q, {}_a^{RC}D_t^\alpha q\right), \xi(t,q)\right] + \mathcal{D}_t^\alpha\left[L\left(t, q, {}_a^{RC}D_t^\alpha q\right) - \alpha\partial_3 L\left(t, q, {}_a^{RC}D_t^\alpha q\right) \cdot {}_a^{RC}D_b^\alpha q, \tau(t,q)\right] = 0, \tag{2}$$

along any fractional Riesz–Caputo extremal q, where ∂_i stands for the partial derivative with respect to the i-th variable.

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http://dx.doi.org/10.1016/j.jmaa.2015.03.060
0022-247X/

Regretfully this result is false as we shall show in the next section.

In order to comprehend Theorem 1.1 let us recall some concepts used in [3].

For $0 < \alpha < 1$, the standard Riemann–Liouville and Caputo (left and right) fractional derivatives are defined by

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\theta)^{-\alpha} f(\theta) d\theta, $$

$$ {}_tD_b^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(-\frac{d}{dt}\right) \int_t^b (\theta-t)^{-\alpha} f(\theta) d\theta, $$

and

$$ {}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\theta)^{-\alpha} f'(\theta) d\theta, $$

$$ {}_t^CD_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\theta-t)^{-\alpha} f'(\theta) d\theta, $$

respectively. Then, the Riesz and the Riesz–Caputo fractional derivatives are defined by

$$ {}_a^RD_b^\alpha f(t) = \frac{1}{2}\left[{}_aD_t^\alpha f(t) - {}_tD_b^\alpha f(t)\right], \tag{3} $$

and

$$ {}_a^{RC}D_b^\alpha f(t) = \frac{1}{2}\left[{}_a^CD_t^\alpha f(t) - {}_t^CD_b^\alpha f(t)\right], \tag{4} $$

respectively.

Definition 1.2. (See [3, Definition 14].) A function q that is a solution of

$$ \partial_2 L\left(t, q(t), {}_a^{RC}D_b^\alpha q(t)\right) - {}_a^RD_b^\alpha \partial_3 L\left(t, q(t), {}_a^{RC}D_b^\alpha q(t)\right) = 0, \quad t \in [a,b], \tag{5} $$

is said to be a fractional Riesz–Caputo extremal for functional (1). Equality (5) is called the fractional Euler–Lagrange equation in the sense of Riesz–Caputo.

Definition 1.3. (See [3, Definition 18].) Given two functions f and g of class C^1 in the interval $[a,b]$, we consider the following operator:

$$ \mathcal{D}_t^\gamma(f,g) = g \cdot {}_a^RD_b^\gamma f + f \cdot {}_a^{RC}D_b^\gamma g, $$

where $t \in [a,b]$ and $\gamma \in \mathbb{R}_0^+$.

Definition 1.4. (See [3, Definition 23].) The functional (1) is said to be invariant under the one-parameter group of infinitesimal transformations

$$ \begin{cases} \bar{t} = t + \varepsilon\tau(t, q(t)) + o(\varepsilon), \\ \bar{q}(t) = q(t) + \varepsilon\xi(t, q(t)) + o(\varepsilon), \end{cases} $$