J. Math. Anal. Appl. 424 (2015) 104–121

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the dynamics of a family of singularly perturbed rational maps

Jianxun Fu, Fei Yang *

Department of Mathematics, Nanjing University, Nanjing, 210093, PR China

ARTICLE INFO

Article history:
Received 20 July 2014
Available online 6 November 2014
Submitted by T. Ransford

Keywords:
Singular perturbation
Cantor circles
Sierpiński carpet

ABSTRACT

In this paper, we study the dynamical behavior of the family of complex rational maps which is given by

$$f_\lambda(z) = \frac{z^n(z^{2n} - \lambda^{n+1})}{z^{2n} - \lambda^{3n-1}},$$

where $n \geq 2$ and $\lambda \in \mathbb{C}^* - \{\lambda : \lambda^{2n-2} = 1\}$. This family of rational maps can be seen as a perturbation of the unicritical polynomial $z \mapsto z^n$ if λ is small. We prove that the Julia set $J(f_\lambda)$ of f_λ is either a quasicircle, a Cantor set of circles, a Sierpiński carpet or a degenerated Sierpiński carpet provided one of the free critical points of f_λ is escaping to the origin or to the infinity. In particular, we prove that there exists suitable λ such that the Julia set $J(f_\lambda)$ is a Cantor set of circles, but f_λ is not topologically conjugate to any McMullen map on their corresponding Julia sets.

1. Introduction

The dynamics of the rational maps are more complicated than the polynomials in general since every polynomial has a completely invariant superattracting basin centered at the infinity. In order to study the topological properties of the Julia sets of polynomials, such as the connectivity, local connectivity, etc., one can construct the puzzles by the external rays and equipotential curves in the basin of infinity. For rational maps, this is impossible in general except for few special examples (see [19] and [16]). However, if the dynamical behaviors of the polynomials are studied well, there is a hope to study the dynamical behaviors of the rational maps nearby well by making a perturbation of the original ones.

The dynamics of the unicritical polynomial $P_n(z) = z^n$, where $n \geq 2$ is completely understood. The dynamical behavior may be not so trivial if one makes a perturbation on P_n. The well known *McMullen map* $F_\lambda(z) = z^n + \lambda/z^d$ was studied extensively in recent years since it produces rich dynamical behaviors

* Corresponding author.
E-mail addresses: jianxunf@163.com (J. Fu), yangfei_math@163.com (F. Yang).

http://dx.doi.org/10.1016/j.jmaa.2014.10.090

and can be seen as a perturbation of $P_n(z)$ (see [4,6,16,15,20] and the references therein). In [6], Devaney, Look and Uminsky gave an Escape Trichotomy Theorem for F_λ according to the iteration numbers of the free critical points before they enter into the immediate attracting basin of the infinity. They proved that the Julia set of F_λ is either a Cantor set, a Cantor set of circles or a Sierpiński carpet if the free critical points of F_λ are attracted by the infinity.

As a variation of McMullen maps, the *McMullen-like maps* $F_{\lambda,\eta}(z) = z^n + \lambda/z^d + \eta$, where $n \geq 2$, $d \geq 1$, also attracted much interest. Xiao, Qiu and Yin gave a topological description of the Julia sets and the Fatou components of $F_{\lambda,\eta}$ according to the dynamical behaviors of the orbits of their free critical points [23]. Garijo and Godillon gave a characterization of subfamily of the McMullen-like mappings by using an arithmetic condition depending only on the polynomial P_n and the pole data [8].

A subset of the Riemann sphere $\widehat{\mathbb{C}}$ is called a *Cantor set of circles* (or *Cantor circles* in short) if it consists of uncountably many closed Jordan curves which is homeomorphic to $\mathcal{C} \times \mathbb{S}^1$, where $\mathcal{C}$ is the Cantor middle third set and $\mathbb{S}^1$ is the unit circle. It is known that the Cantor circles Julia sets can appear in McMullen family and McMullen-like family. McMullen is the first one who found a rational map with this type Julia set and he proved that the Julia set of $f(z) = z^2 + \lambda/z^3$ is a Cantor set of circles if $\lambda \neq 0$ is small enough (see [11, §7]).

The Julia set of some McMullen-like map $F_{\lambda,\eta}$ can be a Cantor set of circles, but the dynamics of $F_{\lambda,\eta}$ in a neighborhood of Julia set is quasiconformally conjugate to that of some McMullen maps (the corresponding Julia sets are also Cantor circles). Therefore, a natural question is to find other types of rational maps whose Julia sets are Cantor circles but the dynamics on the Julia sets are not topologically conjugate to any McMullen map on their corresponding Julia sets. This question was solved in [9] by Haïssinsky and Pilgrim. They proved the existence of such rational maps by quasiconformal surgery. Later, the specific expressions of these rational maps were given in [17].

We are interested in the problem to find a one-dimensional family of such rational maps, such that the Julia sets of some maps in this family are Cantor circles but for each map, there exists essentially only one free critical orbits. For this, we consider the following family of rational maps

$$f_\lambda(z) = \frac{z^n(z^{2n} - \lambda^{n+1})}{z^{2n} - \lambda^{3n-1}}, \tag{1.1}$$

where $n \geq 2$ and $\lambda \in \Lambda := \mathbb{C}^* - \{\lambda : \lambda^{2n-2} = 1\}$. If $n = 1$, $\lambda = 0$ or $\lambda^{2n-2} = 1$, the rational map f_λ degenerates to the polynomial $P_n(z) = z^n$. Therefore, the map f_λ with $\lambda \in \Lambda$ can be seen as a perturbation of the simple polynomial P_n if λ lies in a small neighborhood of the punched points of Λ. This perturbation is essentially different from that of McMullen maps since f_λ not only keeps the dynamics of P_n near the infinity but also keeps the dynamics near the origin.

Note that f_λ has two superattracting fixed points 0 and ∞. We use B_0 and B_∞ to denote the immediate attracting basins of 0 and ∞ respectively. Then $B_0 \cap B_\infty = \emptyset$. The map f_λ has $6n-2$ critical points (counted with multiplicity) since the degree of f_λ is $3n$. The local degrees of 0 and ∞ are both n. Hence, there leaves $4n$ critical points. The forward orbits of 0 and ∞ are trivial since they are fixed by f_λ. We will show in Section 2 that the remaining $4n$ critical points behave symmetrically. So we just have essentially one *free* critical orbit for each f_λ (for the definition of free critical point, see Section 2). This allows us to plot the non-escaping locus of f_λ in $\mathbb{C}$ (see Fig. 2) and give a dynamical classification of the case that the free critical orbits are attracted by 0 and ∞.

1.1. Statement of the main results

According to [22], a *Sierpiński carpet* is a connected, locally connected, nowhere dense compact set which has the property that any two complementary domains are bounded by disjoint Jordan curves. The first