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On moments of classical orthogonal polynomials

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ABSTRACT

In this work, using the inversion coefficients and some connection coefficients between some polynomial sets, we give explicit representations of the moments of all the orthogonal polynomials belonging to the Askey–Wilson scheme. Generating functions for some of these moments are also provided.

1. Introduction

A sequence of polynomials $\{p_n(x)\}$, where $p_n(x)$ is of exact degree n in x, is said to be orthogonal with respect to a Lebesgue–Stieltjes measure $d\alpha(x)$ if

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)d\alpha(x) = 0, \quad m \neq n. \tag{1}$$

Implicit in this definition is the assumption that the moments

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x), \quad n = 0, 1, 2, \ldots, \tag{2}$$

are finite. If the nondecreasing, real-valued, bounded function $\alpha(x)$ also happens to be absolutely continuous with $d\alpha(x) = \rho(x)dx$, $\rho(x) \geq 0$, then (1) and (2) reduce to

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$$\int_{-\infty}^{\infty} p_m(x)p_n(x)\rho(x)dx = 0, \quad m \neq n, \tag{3}$$

and

$$\mu_n = \int_{-\infty}^{\infty} \rho(x)x^n dx, \quad n = 0, 1, 2, \ldots, \tag{4}$$

respectively, and the sequence $\{p_n(x)\}$ is said to be orthogonal with respect to the weight function $\rho(x)$. If, on the other hand, $\alpha(x)$ is a step-function with jumps ρ_j at $x = x_j$, $j = 0, 1, 2, \ldots$, then (1) and (2) take the form of a sum:

$$\sum_{j=0}^{\infty} p_m(x_j)p_n(x_j)\rho_j = 0, \quad m \neq n \tag{5}$$

and

$$\mu_n = \sum_{j=0}^{\infty} x_j^n \rho_j, \quad n = 0, 1, 2, \ldots. \tag{6}$$

The polynomials p_n, in monic form, are given explicitly in terms of the moments by [31]

$$p_n(x) = \frac{1}{d_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix},$$

where

$$d_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \quad \text{under the condition } d_n \neq 0,\ n \geq 0.$$

The previous representation shows that the moments characterize fully the orthogonal family $\{p_n(x)\}$. In [1, p. 295, Theorem 6.3.3] for example, the authors used the moments of the Jacobi polynomials to give the hypergeometric representation of these polynomials.

Note that the classical continuous and discrete orthogonal polynomial families are very much related to probability theory [30] (see also [21]). In the continuous case, the measures of the Hermite, Laguerre and Jacobi polynomials are the normal, the Gamma and the Beta distributions, respectively. In the discrete case, the measures of the Charlier, the Meixner, the Krawtchouk and the Hahn polynomials are the Poisson, the Pascal, the binomial and the hypergeometric distributions. Of course moments play an important role in probability theory and statistics (see [21]).

Despite the important role that the moments play in various topics of orthogonal polynomials and applications to other domains such as statistics and probability theory, no exhaustive repository of moments for the well-known classical orthogonal polynomials can be found in the literature. The book by Koekoek,