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Mean Lipschitz conditions on Bergman space

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A B S T R A C T

For f analytic on the unit disc let $r_t(f)(z) = f(e^{it}z)$ and $f_r(z) = f(rz)$, rotations and dilations respectively. We show that for f in the Bergman space A^p and $0 < \alpha \leq 1$ the following are equivalent.

(i) $\|r_t(f) - f\|_{A^p} = \mathrm{O}(|t|^\alpha)$, $t \to 0$,
(ii) $\|(f')_r\|_{A^p} = \mathrm{O}((1-r)^{\alpha-1})$, $r \to 1^-$,
(iii) $\|f_r - f\|_{A^p} = \mathrm{O}((1-r)^\alpha)$, $r \to 1^-$.

The Hardy space analogues of these conditions are known to be equivalent by results of Hardy and Littlewood and of E. Storozhenko, and in that setting they describe the mean Lipschitz spaces $\Lambda(p, \alpha)$.
On the way, we provide an elementary proof of the equivalence of (ii) and (iii) in Hardy spaces, and show that similar assertions are valid for certain weighted mean Lipschitz spaces.

1. Introduction

Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$. For $1 \leq p \leq \infty$ and $f : \mathbb{D} \to \mathbb{C}$ analytic the integral mean $M_p(r, f)$, $0 \leq r < 1$, is defined as

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{1/p},$$

and

$$M_\infty(r,f) = \max_{-\pi \leq \theta < \pi} \left|f\left(re^{i\theta}\right)\right|.$$

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$M_p(r,f)$ is an increasing function of r. The Hardy space H^p consists of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_p = \sup_{r<1} M_p(r,f) = \lim_{r\to 1^-} M_p(r,f) < \infty.$$

Each f in H^p has radial limits

$$f^*(\theta) = \lim_{r\to 1^-} f\left(re^{i\theta}\right)$$

almost everywhere on $\theta \in [-\pi,\pi]$. The so defined boundary function f^* is p-integrable (essentially bounded if $p=\infty$) and $\|f\|_p$ can be recovered from f^* as $\|f\|_p = (\frac{1}{2\pi}\int_{-\pi}^{\pi}|f^*(\theta)|^p d\theta)^{1/p}$.

For $1 \le p \le \infty$, H^p are Banach spaces. The linear map $f \to f^*$ identifies H^p as the closed subspace of $L^p(-\pi,\pi)$ generated by the set of exponentials $\{e^{in\theta}\}_{n=0}^{\infty}$ or equivalently the subspace consisting of all functions of $L^p(-\pi,\pi)$ whose Fourier series is "of power series type". Additional information for Hardy spaces can be found in [3], and we follow this reference for notation and related material.

For f analytic on $\mathbb{D}$ we write

$$r_t(f)(z) = f\left(e^{it}z\right), \quad t \in \mathbb{R},$$

for the rotated function. If f has boundary values f^* a.e. on $[-\pi,\pi]$ we view f^* extended periodically and we write

$$\tau_t\left(f^*\right)(\theta) = f^*(\theta+t), \quad t \in \mathbb{R}$$

for the translated f^*. It is clear that in this case $r_t(f)^* = \tau_t(f^*)$.

For $f \in H^p$, $p<\infty$, by the continuity of the integral we have $\lim_{t\to 0}\|\tau_t(f^*) - f^*\|_p = 0$. Specifying the rate of this convergence imposes restriction on f.

Definition 1.1. For $1 \le p < \infty$ and $0 < \alpha \le 1$ the analytic mean Lipschitz space $\Lambda(p,\alpha)$ is the collection of $f \in H^p$ such that

$$\left\|\tau_t\left(f^*\right) - f^*\right\|_p \le C|t|^\alpha, \quad -\pi \le t < \pi, \tag{1.1}$$

where C is a constant. The subspace $\lambda(p,\alpha)$ consists of all $f \in H^p$ which satisfy

$$\left\|\tau_t\left(f^*\right) - f^*\right\|_p = \mathrm{o}\left(|t|^\alpha\right), \quad t \to 0. \tag{1.2}$$

Note that these spaces can be defined more generally, as subspaces of $L^p(-\pi,\pi)$, to consist of all L^p functions f that satisfy (1.1) and (1.2) (with f in place of f^*). It was in this general setting that they were first studied in the 1920s by Hardy, Littlewood and others, in connection with convergence and summability of Fourier series, fractional integrals and fractional derivatives, see [7–9]. For the main properties of $\Lambda(p,\alpha)$ see [2,5,13]. Among several other results Hardy and Littlewood proved the following theorem.

Theorem A. *Suppose $1 \le p < \infty$, $0 < \alpha \le 1$ and $f \in H^p$. Then the following are equivalent*

(a) $f \in \Lambda(p,\alpha)$,
(b) $M_p(r,f') = \mathrm{O}((1-r)^{\alpha-1})$, $r \to 1^-$.

Note that if an analytic function satisfies (b) for some $\alpha > 0$, then it belongs in H^p.

For f analytic on $\mathbb{D}$ let