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Minimal immersions of Riemannian manifolds in products of space forms

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ABSTRACT

In this paper, we give natural extensions to cylinders and tori of a classical result due to T. Takahashi [8] about minimal immersions into spheres. More precisely, we deal with Euclidean isometric immersions whose projections in $\mathbb{R}^N$ satisfy a spectral condition of their Laplacian.

1. Introduction

An isometric immersion $f : M^m \to N^n$ of a Riemannian manifold M in another Riemannian manifold N is said to be *minimal* if its mean curvature vector field H vanishes. The study of minimal surfaces is one of the oldest subjects in differential geometry, having its origin with the work of Euler and Lagrange. In the last century, a series of works have been developed in the study of properties of minimal immersions, whose ambient space has constant sectional curvature. In particular, minimal immersions in the sphere $\mathbb{S}^n$ play an important role in the theory, as for example the famous paper of J. Simons [7].

Let $f : M^m \to \mathbb{R}^n$ be an isometric immersion of an m-dimensional manifold M into the Euclidean space $\mathbb{R}^n$. Associated with the induced metric on M, it is defined the Laplace operator Δ acting on $C^\infty(M)$. This Laplacian can be extended in a natural way to the immersion f. A well-known result by J. Eells and J.H. Sampson [3] asserts that the immersion f is minimal if and only if $\Delta f = 0$. The following result, due to T. Takahashi [8], states that the immersion f *realizes* a minimal immersion in a sphere if and only if its coordinate functions are eigenfunctions of the Laplace operator with the same nonzero eigenvalue.

Theorem 1. *Let $F : M^m \to \mathbb{R}^{n+1}$ be an isometric immersion such that*

$$\Delta F = -mcF$$

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for some constant $c \neq 0$. Then $c > 0$ and there exists a minimal isometric immersion $f : M^m \to \mathbb{S}_c^n$ such that $F = i \circ f$.

O. Garay generalized Theorem 1 for the hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ satisfying $\Delta f = Af$, where A is a constant $(n+1) \times (n+1)$ diagonal matrix. He proved in [4] that such a hypersurface is either minimal or an open subset of a sphere or of a cylinder. In this direction, J. Park [6] classified the hypersurfaces in a space form or in Lorentzian space whose immersion f satisfies $\Delta f = Af + B$, where A is a constant square matrix and B is a constant vector. Similar results were obtained in [1], where the authors study and classify pseudo-Riemannian hypersurfaces in pseudo-Riemannian space forms which satisfy the condition $\Delta f = Af + B$, where A is an endomorphism and B is a constant vector. We would point out that this problem is strongly connected to other related topics as immersions whose mean curvature satisfies a polynomial equation in the Laplacian, biharmonic submanifolds, finite type submanifolds and others. For the last, we refer to [2], where the author discusses the problem of determining the geometrical structure of a submanifold knowing some simple analytic information.

In this work we shall deal with an isometric immersion $f : M^m \to \mathbb{R}^N$ of a Riemannian manifold M^m into the Euclidean space R^N. If the submanifold $f(M)$ is contained in a cylinder $\mathbb{S}_c^n \times \mathbb{R}^k \subset \mathbb{R}^N$ or in a torus $\mathbb{S}_c^n \times \mathbb{S}_d^k \subset \mathbb{R}^N$, we shall call that the *immersion f realizes an immersion in a cylinder* or in a *torus*, respectively. Motivated by recent works on the submanifold theory in the product of space forms [5], we obtain theorems that give us necessary and sufficient conditions for an isometric immersion $f : M^m \to \mathbb{R}^N$ to realize a minimal immersion in a cylinder or in a torus (cf. Theorems 4 and 8).

2. Preliminaries

Let M^m be a Riemannian manifold and $h \in C^\infty(M)$. The *hessian* of h is the symmetric section of $\mathrm{Lin}(TM \times TM)$ defined by

$$\operatorname{Hess} h(X, Y) = XY(h) - \nabla_X Y(h)$$

for all $X, Y \in TM$. Equivalently,

$$\operatorname{Hess} h(X, Y) = \langle \nabla_X \operatorname{grad} h, Y \rangle$$

where $X, Y \in TM$ and $\operatorname{grad} h$ is the gradient of h. The *Laplacian* Δh of a function $h \in C^\infty(M)$ at the point $p \in M$ is defined as

$$\Delta h(p) = \operatorname{trace} \operatorname{Hess} h(p) = \operatorname{div} \operatorname{grad} h(p).$$

Consider now an isometric immersion $f : M^m \to \mathbb{R}^n$. For a fixed $v \in \mathbb{R}^n$, let $h \in C^\infty(M)$ be the height function with respect to the hyperplane normal to v, given by $h(p) = \langle f(p), v \rangle$. Then

$$\operatorname{Hess} h(X, Y) = \langle \alpha_f(X, Y), v \rangle \tag{1}$$

for any $X, Y \in TM$. For an isometric immersion $f : M^n \to \mathbb{R}^n$, by $\Delta f(p)$ at the point $p \in M$ we mean the vector

$$\Delta f(p) = \big(\Delta f_1(p), \ldots, \Delta f_n(p)\big),$$

where $f = (f_1, \ldots, f_n)$. Taking traces in (1) we obtain