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Doubly optimal homogeneous trace Sobolev inequality in a solid torus

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ABSTRACT

In this paper, using techniques that exploit the symmetry presented by the solid torus we study the problem of determining both the best constants for the doubly homogeneous trace Sobolev inequality, which in the general case studied by Maggi and Villani in [16].

1. Overview and introduction to the topic

Sobolev inequalities are among the most important functional inequalities in analysis because of their very interesting autonomous existence and also because of their strong connection with the solvability of a large number of nonlinear partial differential equations. They express a strong integrability and/or regularity property for a function f in terms of some integrability property for some derivatives of f. In this paper, we study the doubly optimal homogeneous trace Sobolev inequality in a solid torus.

1.1. Sobolev inequalities

We are starting this part with the following question:

`How can one control the size of a function in terms of the size of its gradient?`

On the real line the answer is given by a useful calculus inequality. Namely, for any smooth function f with compact support on the line,

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$$\left|f(x)\right| \leqslant \frac{1}{2}\int_{-\infty}^{+\infty}\left|f'(t)\right|dt. \tag{1}$$

The factor $\frac{1}{2}$ in the inequality (1) comes from the fact that f vanishes at both $-\infty$ and ∞.

Furthermore, in 1896 Steklov [22] proved that the inequality

$$\int_0^l f^2(x)dx \leqslant \left(\frac{1}{\pi}\right)^2\int_0^l \left|f'(x)\right|^2 dx \tag{2}$$

holds for all functions which are continuously differentiable on $[0,l]$ and they have zero mean there. The inequality (2) was among earliest inequalities with sharp constant that appeared in mathematical physics. The fact that the constant in (2) is sharp was emphasized by Steklov in [21].

Next year (1897), Steklov published the article [23], in which the following analogue of the inequality (2) was proved:

$$\int_\Omega f^2 dx \leqslant C\int_\Omega |\nabla f|^2 dx \tag{3}$$

Here, ∇ stands for the gradient operator and the integral on the right-hand side is called the Dirichlet integral. The assumptions made by Steklov are as follows: Ω is a bounded three-dimensional domain whose boundary is piecewise smooth and f is a real C^1-function on $\bar{\Omega}$ vanishing on $\partial\Omega$. Again, the inequality (3) was obtained by Steklov with the sharp constant equal to λ_1^{-1}, where λ_1 is the smallest eigenvalue of the Dirichlet Laplacian in Ω.

It is natural to wonder if there is such an inequality for smooth compactly supported functions in higher-dimensional Euclidean spaces. Namely the focus is on the direct generalizations of inequalities (2) and (3) on an arbitrary $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 1$, that is, to inequalities of the following form:

$$\left(\int_\Omega |f|^q dx\right)^{\frac{1}{q}} \leqslant C\left(\int_\Omega |\nabla f|^p dx\right)^{\frac{1}{p}} \tag{4}$$

Clearly the question is what are the appropriate conditions to be satisfied by n, p and q such that the inequality (4) holds for all smooth compactly supported functions in Ω?

The answer to this question was first addressed in this form by Sobolev in 1938 [20] in the case where $\Omega = \mathbb{R}^n$. It is positive if and only if

$$q = p^* = \frac{np}{n-p},$$

and this exponent is called and it is **critical** in the sense that it cannot become lower nor higher and the inequality be in effect. More precisely, the Sobolev embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ asserts that for any $p \in [1,n)$ exists a positive constant $K(n,p)$ such that for every $f \in W^{1,p}(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |f|^{p^*} dx\right)^{\frac{1}{p^*}} \leqslant K(n,p)\left(\int_{\mathbb{R}^n} |\nabla f|^p dx\right)^{\frac{1}{p}}, \tag{5}$$

where p^* is defined as above, and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$.