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Two variable extensions of the Laguerre and disc polynomials

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ABSTRACT

This work contains a detailed study of a one parameter generalization of the $2D$-Hermite polynomials and a two parameter extension of Zernike's disc polynomials. We derive linear and bilinear generating functions, and explicit formulas for our generalizations and study integrals of products of some of these $2D$ orthogonal polynomials. We also establish a combinatorial inequality involving elementary symmetric functions and solve the connection coefficient problem for our polynomials.

1. Introduction

The $2D$-Hermite polynomials

$$H_{m,n}(z_1, z_2) = \sum_{k=0}^{m\wedge n} \binom{m}{k}\binom{n}{k}(-1)^k k! z_1^{m-k} z_2^{n-k} \tag{1.1}$$

were introduced by Ito in [16] and have many applications to physical problems, see [2,4,22,25–27]. Mathematical properties of these polynomials have been developed in [7–9]. A multilinear generating function, of Kibble–Slepian type [17], is proved in [11]. The combinatorics of integrals of products of $2D$-Hermite polynomials has been explored in [13] while the combinatorics of the $2D$-Hermite polynomials, of their generating functions including the Kibble–Slepian type formula is in our forthcoming paper, [14], Ismail

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and Zhang [15] gave two q-analogues of the $2D$-Hermite polynomials. They studied these q-polynomials in great detail.

Ismail and Zhang [15] identified a general class of two variable polynomials whose measure is the product of the uniform measure on the circle times a radial measure. This class not only contains the $2D$-Hermite polynomials and their q-analogues but it also contains the generalized Zernike (or disc) polynomials and their q-analogues. This will be formulated in Section 2. The generalized disc polynomials have been known for a long time, see [20,19,18]. More recent papers are [28] and [1]. They form a one parameter generalization of the original Zernike polynomials.

In Section 3 we study a one parameter extension of the $2D$-Hermite polynomials. These polynomials appeared in [15] and are denoted by $Z^{(\beta)}_{m,n}(z_1, z_2)$. We record the definition, orthogonality relation, and the three term recurrence relations in Section 3. In Section 3 we also derive several differential properties of our polynomials. Section 4 contains a two parameter generalization of Zernike polynomials, so they contain one additional parameter. Several authors considered the sign regularity of integrals of products of orthogonal polynomials times certain functions. Some of the literature on this problem is in Askey's classic [3], see also Chapter 9 of [10]. In Section 5 we analyze the positivity of the integrals

$$\int_0^\infty \prod_{j=1}^N L^{(\alpha-n_j)}_{n_j}(-x)e^{-\lambda x}dx.$$

Our analysis leads to a curious rationale symmetric functions with nonnegative integral coefficients. This will be stated as Theorem 5.6.

2. General construction

The general construction given here for $2D$-systems is due to Ismail and Zhang [15]. One starts with a system of orthogonal polynomials $\{\phi_n(r;\alpha)\}$ satisfying the orthogonality relation

$$\int_0^\infty \phi_m(r;\alpha)\phi_n(r;\alpha)r^\alpha d\mu(r) = \zeta_n(\alpha)\delta_{m,n}, \quad \alpha \ge 0. \tag{2.1}$$

It is assumed the μ does not depend on α. Let

$$\phi_n(r;\alpha) = \sum_{j=0}^n c_j(n,\alpha)r^{n-j}, \quad c_j(n,\alpha) \in \mathbb{R}, \tag{2.2}$$

and define polynomials

$$f_{n+\alpha,n}(z_1,z_2) = \sum_{j=0}^n c_j(n,\alpha)z_1^{n+\alpha-j}z_2^{n-j} = z_1^\alpha \phi_n(z_1z_2;\alpha). \tag{2.3}$$

Also define $f_{n,m}(z_1,z_2) = f_{m,n}(z_2,z_1)$. Thus $\overline{f_{m,n}(z,\bar{z})} = f_{n,m}(z,\bar{z})$.

Theorem 2.1. *For $m \ge n$ the polynomials $\{f_{m,n}(z,\bar{z})\}$ satisfy the orthogonality relation*

$$\int_{\mathbb{R}^2} f_{m,n}(z,\bar{z})\overline{f_{s,t}(z,\bar{z})}\frac{d\theta}{2\pi}d\mu(r^2) = \zeta_n(m-n)\delta_{m,s}\delta_{n,t}. \tag{2.4}$$

We now come to the three term recurrence relations.