J. Math. Anal. Appl. 424 (2015) 344–360

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mixed estimates for degenerate multi-linear operators associated to simplexes

Robert Kesler

Department of Mathematics, Cornell University, 212 Garden Avenue, Ithaca, NY 14853, United States

A R T I C L E I N F O

Article history:
Received 9 February 2014
Available online 7 November 2014
Submitted by M. Peloso

Keywords:
Multi-linear integrals
Mixed estimates
Vector-valued inequalities

A B S T R A C T

We prove that the degenerate trilinear operator $C_3^{-1,1,1}$ given by the formula

$$C_3^{-1,1,1}(f_1, f_2, f_3)(x) = \int\limits_{x_1<x_2<x_3} \hat{f}_1(x_1)\hat{f}_2(x_2)\hat{f}_3(x_3)e^{2\pi ix(-x_1+x_2+x_3)}dx_1dx_2dx_3$$

satisfies for every (p_1, p_2, p_3) such that $2 < p_1 \leq \infty$, $1 < p_2, p_3 < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} < 1$, and $\frac{1}{p_2} + \frac{1}{p_3} < 3/2$, the mixed estimate

$$\left\|C_3^{-1,1,1}(f_1, f_2, f_3)\right\|_{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}} \lesssim_{p_1,p_2,p_3} \|\hat{f}_1\|_{p_1'}\|f_2\|_{p_2}\|f_3\|_{p_3}$$

for all $f_1 \in L^{p_1}(\mathbb{R}) : \hat{f}_1 \in L^{p_1'}(\mathbb{R})$, $f_2 \in L^{p_2}(\mathbb{R})$, and $f_3 \in L^{p_3}(\mathbb{R})$. Mixed estimates for some generalizations of $C_3^{-1,1,1}$ are also established.

1. Introduction

Many boundedness results have been obtained for singular multi-linear integrals with nonclassical symbols, see e.g. [3,5–9,11,13,14]. One such example is a theorem due to Christ and Kiselev, which states that the bilinear operator $\tilde{C}_2^{\alpha_1,\alpha_2}$ initially defined on $L^1(\mathbb{R})$ functions by

$$\tilde{C}_2^{\alpha_1,\alpha_2}(f_1, f_2)(x) = \int\limits_{x_1<x_2} f_1(x_1)f_2(x_2)e^{2\pi ix(\alpha_1x_1+\alpha_2x_2)}dx_1dx_2$$

E-mail address: rkesler@math.cornell.edu.

http://dx.doi.org/10.1016/j.jmaa.2014.11.005
0022-247X/

extends to a continuous map from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ into $L^{\frac{p_1' p_2'}{p_1'+p_2'}}(\mathbb{R})$, assuming $1 \le p_1, p_2 < 2$ and $\alpha_1, \alpha_2 \neq 0$, see [1,7]. Lacey and Thiele proved a wide range of L^p estimates in [5] for a related operator called the bilinear Hilbert transform given by the formula

$$BHT(f_1, f_2)(x) = \tilde{C}_2^{1,1}(\hat{f}_1, \hat{f}_2)(x) = \int\limits_{x_1<x_2} \hat{f}_1(x_1)\hat{f}_2(x_2)e^{2\pi ix(x_1+x_2)}dx_1dx_2,$$

after which boundedness was shown by Muscalu, Tao, and Thiele [9] for a trilinear variant of the BHT called the Biest, which takes the form

$$C_3^{1,1,1}(f_1, f_2, f_3)(x) = \int\limits_{x_1<x_2<x_3} \hat{f}_1(x_1)\hat{f}_2(x_2)\hat{f}_3(x_3)e^{2\pi ix(x_1+x_2+x_3)}dx_1dx_2dx_3.$$

However, multi-linear integrals with sign degeneracies such as the operator

$$C_3^{-1,1,1}(f_1, f_2, f_3)(x) = \int\limits_{x_1<x_2<x_3} \hat{f}_1(x_1)\hat{f}_2(x_2)\hat{f}_3(x_3)e^{2\pi ix(-x_1+x_2+x_3)}dx_1dx_2dx_3$$

are known to satisfy no L^p estimates, see [7]. Despite this fact, we prove in Theorem 3 that there exists a constant C_{p_1,p_2,p_3} such that for all $f_1 \in L^{p_1}(\mathbb{R})$ satisfying $\hat{f}_1 \in L^{p_1'}(\mathbb{R})$ along with $f_2 \in L^{p_2}(\mathbb{R})$ and $f_3 \in L^{p_3}(\mathbb{R})$,

$$\left\|C_3^{-1,1,1}(f_1, f_2, f_3)\right\|_{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}} \le C_{p_1,p_2,p_3}\|\hat{f}_1\|_{p_1'}\|f_2\|_{p_2}\|f_3\|_{p_3}$$

as long as $2 < p_1 \le \infty, 1 < p_2, p_3 < \infty, \frac{1}{p_1} + \frac{1}{p_2} < 1$ and $\frac{1}{p_2} + \frac{1}{p_3} < 3/2$. We also establish mixed boundedness for $C_5^{1,1,-1,1,1}$ in Theorem 5 before handling $C_8^{1,1,-1,1,1-1,1,1}$ in Theorem 6 and the main conclusion in this paper, namely Theorem 7, which establishes mixed boundedness of the generalized n-linear integral

$$C_n^{\vec{\epsilon}}(f_1, ..., f_n)(x) = \int\limits_{x_1<...<x_n} \hat{f}_1(x_1)...\hat{f}_n(x_n)e^{2\pi ix(\vec{\epsilon}\cdot\vec{x})}d\vec{x}, \quad \vec{\epsilon} \in \{\pm 1\}$$

for a large range of exponents and answers a question posed by C. Muscalu. The proofs rely on the Christ–Kiselev martingale structure decomposition from [1], in addition to a generalized version of the Littlewood–Paley inequality of Rubio de Francia for L^p functions with $p < 2$, see Rubio de Francia [12] and Lacey [4], and a maximal l^2 vector-valued inequality for the bilinear Hilbert transform and its generalizations, see Lemma 2.

2. Mixed estimates

2.1. Preliminaries

To introduce the martingale structure decomposition of Christ and Kiselev, we prove continuity for the map $\tilde{C}_2^{\alpha_1,\alpha_2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{\frac{p_1' p_2'}{p_1'+p_2'}}(\mathbb{R})$ given by

$$\tilde{C}_2^{\alpha_1,\alpha_2}(f_1, f_2)(x) = \int\limits_{x_1<x_2} f_1(x_1)f_2(x_2)e^{2\pi ix(\alpha_1x_1+\alpha_2x_2)}dx_1dx_2$$