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Uniqueness for the heat equation in Riemannian manifolds

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ABSTRACT

We investigate uniqueness, in suitable weighted Lebesgue spaces, of solutions to the heat equation in geodesically complete Riemannian manifolds.

1. Introduction

We are concerned with uniqueness of solutions to the following parabolic Cauchy problem:

$$\begin{cases} \partial_t u = \Delta u + f & \text{in } M \times (0,T] =: S_T \\ u = u_0 & \text{in } M \times \{0\}, \end{cases} \tag{1.1}$$

where M is a geodesically complete N-dimensional Riemannian manifold, Δ denotes the Laplace–Beltrami operator on M, $f \in C(S_T)$, $u_0 \in L^p_{loc}(M)$ for some $p \in [1,2]$. Clearly, uniqueness for problem (1.1) follows, if it is shown that the unique solution to problem

$$\begin{cases} \partial_t u = \Delta u & \text{in } S_T \\ u = 0 & \text{in } M \times \{0\} \end{cases} \tag{1.2}$$

is $u \equiv 0$. Uniqueness of bounded solutions to problem (1.2) has been largely investigated in the literature. Furthermore (see, e.g., [5, Theorem 6.2]), it is well known that it is equivalent to *stochastic completeness* of M; this means that

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$$\int_M p(x,y,t)d\mu(y) = 1 \quad \text{for all } x \in M,\ t > 0,$$

where p is the *heat kernel* and $d\mu$ the Riemannian volume element.

In particular, in [15] it is proved that any geodesically complete Riemannian manifold M with Ricci curvature bounded from below is stochastically complete. Then the same result has been extended in [9] and in [7] to allow the Ricci curvature of M to be also negative, provided it satisfies proper bounds from below. Under similar hypotheses, in [10] uniqueness in $L^1(M)$ is established. In order to describe other related results, we need to introduce some notations. For any $x_0 \in M$ and $R > 0$ let $B_R(x_0) := \{x \in M \,|\, d(x,x_0) < R\}$, where $d(x,x_0)$ is the geodesic distance between x and x_0. Furthermore, let $V(x_0,r)$ denote the Riemannian volume of $B_R(x_0)$. In [2] it is proved that if $\log V(x_0,R) = o(R)$ as $R \to \infty$, then M is stochastically complete. Moreover, in [4] it is stated that if M is a geodesically complete Riemannian manifold, and if, for some point $x_0 \in M$,

$$\int_1^\infty \frac{r}{\log V(x_0,r)}dr = \infty,$$

then M is stochastically complete. We refer the reader to [5] for a complete account on results about this subject.

Observe that the previous result from [4] (see also [5, Theorem 9.1]) is obtained as a consequence of the following more general result. Indeed, in [5, Theorem 9.2] it is proved that if $u \in C^2(M \times (0,T])$ is a solution to problem (1.2), with the initial condition understood in the sense of $L^2_{loc}(M)$, then $u \equiv 0$ in S_T, provided that, for some $x_0 \in M$ and $R_0 > 0$, there holds

$$\int_0^T \int_{B_R(x_0)} u^2(x,t)d\mu(x)dt \le e^{\varphi(R)} \quad \text{for all } R > R_0, \tag{1.3}$$

φ being a positive increasing continuous function defined in $(0,\infty)$ such that

$$\int_{R_0}^\infty \frac{r}{\varphi(r)}dr = \infty. \tag{1.4}$$

Note that from this result we can immediately deduce the following uniqueness result of solutions to problem (1.2) belonging to a suitable Lebesgue weighted space. In fact, for any $p \ge 1$, $g \in C(M)$, $g > 0$ in M, let

$$L^p_g\big(M \times (0,T)\big) := \left\{ u : S_T \to \mathbb{R} \text{ measurable} \,\Big|\, \int_0^T \int_M \big|u(x,t)\big|^p g(x)d\mu(x)dt < \infty \right\}.$$

Suppose that $u \in L^2_g(M \times (0,T))$, where

$$g(x) := e^{-\varphi(d(x,x_0))} \quad (x \in M),$$

for some $x_0 \in M$ and φ as above. Let $C := \|u\|_{L^2_g(S_T)}$. Then, since φ is increasing, for all $R > R_0$,