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Second order estimates for boundary blowup solutions of quasilinear elliptic equations

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ABSTRACT

Let $\Omega \subset R^N$ be a bounded smooth domain. We investigate the effect of the mean curvature of the boundary $\partial\Omega$ on the behavior of the blow-up solution to the equation $\Delta u = u^p|\nabla u|^q$. Under appropriate conditions on p and q, we find asymptotic expansions up to the second order of the solution u in terms of the distance from x to the boundary $\partial\Omega$.

1. Introduction

We study the boundary blow-up problem

$$\Delta u = u^p|\nabla u|^q \quad \text{in } \Omega,\ u \to \infty \text{ as } x \to \partial\Omega, \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, $p > 0$, $0 \leq q \leq (p+3)/(p+2)$ and $p+q > 1$. To prove the existence of a positive large solution we first consider, for $0 < \epsilon < 1$, the problem

$$\Delta u = u^p\left(\epsilon + |\nabla u|^2\right)^{\frac{q}{2}} \quad \text{in } \Omega,\ u \to \infty \text{ as } x \to \partial\Omega.$$

The existence of a positive solution $u = u_\epsilon$ for this new problem is proved in [3,6,12]. By Lemma 4.1 of [12] we know that, for any compact $G \subset \Omega$ there is a constant M such that $u_\epsilon(x) \leq M$ for all $x \in G$. Note that, since $|\nabla u|^q < (\epsilon+|\nabla u|^2)^{\frac{q}{2}} < (1+|\nabla u|^2)^{\frac{q}{2}}$, the constant M can be chosen independent of ϵ. Moreover, by Theorem 3.1 (p. 266) of [8] (or by Theorems 14.1 and 15.1 of [7]) we get a bound of $|\nabla u_\epsilon(x)|$ in terms of M for all $x \in G$. Finally, by Theorem 4.2 of [12] a sequence u_{ϵ_i}, with $\epsilon_i \to 0$, tends to a solution u of problem (1).

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We are interested in the behavior of the solution u near the boundary $\partial\Omega$. Problems of this kind are discussed in many papers; see, for instance, [1,2,4,5,10,13] and the survey paper [14]. For $q = 0$, C. Bandle in [1] has found the estimate

$$u(x) = \left(\frac{p-1}{\sqrt{2(p+1)}}\delta(x)\right)^{\frac{2}{1-p}}\left[1 + \frac{(N-1)H(\bar{x})}{p+3}\delta(x) + o\big(\delta(x)\big)\right], \tag{2}$$

where $\delta(x)$ denotes the distance from x to the boundary $\partial\Omega$, and $H(\bar{x})$ denotes the mean curvature of $\partial\Omega$ at the point $\bar{x}$ nearest to x.

In the present paper we find an estimate similar to (2) for the solution of problem (1). More precisely, for $0 \le q < (p+3)/(p+2)$ we find

$$u(x) = \phi\big(\delta(x)\big)\left[1 + \frac{(2-q)(N-1)H(x)}{2(p+3-q(p+2))}\delta(x) + O(1)\big(\delta(x)\big)^{\sigma}\right], \tag{3}$$

where

$$\phi(t) = \left(\frac{2-q}{p+q-1}\right)^{\frac{2-q}{p+q-1}}\left(\frac{p+1}{2-q}\right)^{\frac{1}{p+q-1}} t^{\frac{q-2}{p+q-1}}, \tag{4}$$

$(N-1)H(x) = -\Delta\delta(x)$, $\sigma > 1$ is a suitable constant and $O(1)$ is a bounded quantity. Of course, we can replace $H(x)$ by $H(\bar{x})$ as in (1), replacing $O(1)(\delta(x))^{\sigma}$ by $o(\delta(x))$.

The case $q = (p+3)/(p+2)$ appears to be critical. In this case we find

$$u(x) = \phi\big(\delta(x)\big)\left[1 + \frac{(N-1)H(x)}{2(p+1)}\delta(x)\log\frac{1}{\delta(x)} + O(1)\delta(x)\left(\log\frac{1}{\delta(x)}\right)^{\sigma}\right], \tag{5}$$

where $0 < \sigma < 1$.

2. Main results

Let $p > 0$, $0 \le q < 2$, $p + q > 1$. Consider the equation in (1) in dimension $N = 1$ and $\Omega = (0, \infty)$. If $u = \phi(t) > 0$ and $\phi'(t) < 0$ we have

$$\phi'' = \phi^p\big(-\phi'\big)^q. \tag{6}$$

A solution of (6) such that $\phi(t) \to \infty$ as $t \to 0$ is precisely the function defined in (4).

In what follows we denote by $C > 1$ a constant which may change from term to term.

Lemma 2.1. *Let $A(\rho, R) \subset \mathbb{R}^N$, $N \ge 2$, be the annulus with radii ρ and R centered at the origin. Let ϕ be the function defined in (4), let $u(x)$ be a radial solution to problem (1) in $\Omega = A(\rho, R)$, and let $v(r) = u(x)$ for $r = |x|$. If $p > 0$, $0 \le q < (p+3)/(p+2)$ and $p + q > 1$ we have*

$$v(r) < \phi(R-r)\big[1 + C(R-r)\big], \quad r \in \big(r', R\big), \tag{7}$$

$$v(r) > \phi(r-\rho)\big[1 - C(r-\rho)\big], \quad r \in \big(\rho, r''\big). \tag{8}$$