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Rigidity of proper holomorphic self-mappings of the pentablock

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A R T I C L E I N F O

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A B S T R A C T

The pentablock is a Hartogs domain in $\mathbb{C}^3$ over the symmetrized bidisc in $\mathbb{C}^2$. The domain is a bounded inhomogeneous pseudoconvex domain, which does not have a $\mathcal{C}^1$ boundary. Recently, Agler–Lykova–Young constructed a special subgroup of the group of holomorphic automorphisms of the pentablock, and Kosiński fully described the group of holomorphic automorphisms of the pentablock. The aim of the present study is to prove that any proper holomorphic self-mapping of the pentablock must be an automorphism.

1. Introduction

Let $\mathbb{C}^{2\times 2}$ denote the space of 2×2 complex matrices, with the usual operator norm, i.e., for a matrix $A \in \mathbb{C}^{2\times 2}$,

$$\|A\| := \sup\big\{\|zA\|/\|z\| : z \in \mathbb{C}^2,\ z \neq 0\big\},$$

where $\mathbb{C}^2$ is equipped with the standard Hermitian norm. Recently, Agler, Lykova and Young [2] introduced the bounded domain $\mathcal{P}$ by

$$\mathcal{P} := \big\{(a_{21}, \operatorname{tr} A, \det A) : A = [a_{ij}]_{i,j=1}^2 \in \mathbb{B}\big\},$$

where

$$\mathbb{B} := \big\{A \in \mathbb{C}^{2\times 2} : \|A\| < 1\big\}$$

denotes the open unit ball in the space $\mathbb{C}^{2\times 2}$ with the usual operator norm. Thus, $\mathcal{P}$ is an image of $\mathbb{B}$ under the holomorphic mapping $A = [a_{ij}] \mapsto (a_{21}, \operatorname{tr} A, \det A)$. The domain $\mathcal{P}$ is called the *pentablock* by Agler,

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Lykova and Young [2] because $\mathcal{P} \cap \mathbb{R}^3$ is a convex body bounded by five faces, in which three of them are flat and two are curved.

The pentablock $\mathcal{P}$ is polynomially convex and starlike about the origin, but neither circled nor convex, and it does not have a $\mathcal{C}^1$ boundary (see Agler, Lykova and Young [2]). The pentablock is a bounded inhomogeneous domain (see Th. 15 in Kosiński [14]). See Agler, Lykova and Young [2] and Kosiński [14] for details of the complex geometry and function theory of the pentablock $\mathcal{P}$.

The pentablock $\mathcal{P}$ arises in connection with the *structured singular value*, which is a cost function on matrices introduced by control engineers in the context of robust stabilization with respect to modeling uncertainty (e.g., Doyle [8]). The structured singular value is denoted by μ, and engineers have proposed an interpolation problem called the *μ-synthesis problem*, which arises from this source. Attempts to solve cases of this interpolation problem have also led to the study of two other domains: the *symmetrized disc* (e.g., [3,11]) and the *tetrablock* (e.g., [10,20]), in $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively, which have many properties of interest to specialists in several complex variables (e.g., [1,11,13]) and for operator theorists (e.g., [5,16]).

Throughout this study, $\mathbb{D}$ denotes the open unit disc in the complex plane, while $\mathbb{T}$ denotes the unit circle.

The pentablock is closely related to the symmetrized bidisc $\mathbb{G}_2$, which is a bounded domain in $\mathbb{C}^2$, as follows:

$$\mathbb{G}_2 := \big\{(\lambda_1 + \lambda_2, \lambda_1\lambda_2) \in \mathbb{C}^2 : \lambda_1, \lambda_2 \in \mathbb{D}\big\}.$$

For $(s,p) \in \mathbb{C}^2$, it is easy to check (see Th. 2.1 in [2]) that $(s,p) \in \mathbb{G}_2$ if and only if $|s - \bar{s}p| + |p|^2 < 1$. Thus, the symmetrized bidisc $\mathbb{G}_2$ can also be described as

$$\mathbb{G}_2 = \big\{(s,p) \in \mathbb{C}^2 : |s - \bar{s}p| + |p|^2 < 1\big\}.$$

The symmetrized bidisc is important because it is the first known example of a bounded pseudoconvex domain for which the Carathéodory and Lempert functions coincide, but which cannot be exhausted by domains biholomorphic to convex ones (see Costara [6] and Edigarian [9]).

Edigarian and Zwonek [11] provided characterizations of proper holomorphic self-mappings of the symmetrized polydisc, which reduces the result as follows in the special case of the symmetrized bidisc.

Theorem 1. *(See Edigarian and Zwonek [11].) Let $f : \mathbb{G}_2 \to \mathbb{G}_2$ be a holomorphic mapping. Then, f is proper if and only if there exists a finite Blaschke product B such that*

$$f(\lambda_1 + \lambda_2, \lambda_1\lambda_2) = \big(B(\lambda_1) + B(\lambda_2), B(\lambda_1)B(\lambda_2)\big) \quad (\lambda_1, \lambda_2 \in \mathbb{D}).$$

In particular, f is an automorphism if and only if

$$f(\lambda_1 + \lambda_2, \lambda_1\lambda_2) = \big(\nu(\lambda_1) + \nu(\lambda_2), \nu(\lambda_1)\nu(\lambda_2)\big) \quad (\lambda_1, \lambda_2 \in \mathbb{D}),$$

where ν is an automorphism of the unit disc $\mathbb{D}$. Because $\mathrm{Aut}(\mathbb{G}_2)$ *does not act transitively on $\mathbb{G}_2$, the symmetrized bidisc is inhomogeneous.*

By the definition of the domain $\mathcal{P}$, the pentablock is a Hartogs domain in $\mathbb{C}^3$ over the symmetrized bidisc $\mathbb{G}_2$. Indeed, it is clear from the definition that $\mathcal{P}$ is fibered over $\mathbb{G}_2$ by the map

$$(a, s, p) \mapsto (s, p)$$

because, if $A \in \mathbb{B}(\subset \mathbb{C}^{2\times 2})$, then the eigenvalues of A lie in $\mathbb{D}$ and thus $(\operatorname{tr} A, \det A) \in \mathbb{G}_2$. More precisely (e.g., see Th. 1.1 in Agler, Lykova and Young [2]), we have