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A Riemann–Hilbert boundary value problem in a triangle

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ABSTRACT

In this article a Riemann–Hilbert boundary value problem on an isosceles orthogonal triangle is considered. Using explicit Schwarz–Poisson-type formulae for the triangle, Schwarz-type and Pompeiu-type operators are obtained. Boundary behaviors of these operators are discussed in detail. Finally, we investigate the Riemann–Hilbert boundary value problem for both homogeneous and inhomogeneous Cauchy–Riemann equations. An explicit solvability of the problem is obtained.

1. Introduction

The Riemann–Hilbert-type boundary value problems, its counterparts, and related problems are extensively investigated, see, e.g. [1–8,13–23,28–30,32,35–40]. Besides its theoretical significance, the study of the Riemann–Hilbert problems has many applications in a vast variety of applied sciences, among these applications one may mention crack theory, diffraction theory, elasticity theory, Hele-Shaw flow, hydrodynamics, orthogonal polynomials, and tomography theory [9–12,24,26,27,31,33,34]. Other classical problems such as Dirichlet, Neumann, Schwarz, Robin, and Riemann–Hilbert–Poincaré are special cases or related problems to the Riemann–Hilbert boundary value problem. In the recent years, those problems have been investigated for classes of complex partial differential equations (PDEs) in some special domains, such as the unit disc, half plane, circular ring, quarter plane, quarter ring, half disc and half ring, hyperbolic strips, lens and lune, triangles, and sectors [1,3,6,7,15,17,18,21,22,36,38,39].

In this article, we discuss the solvability of the Riemann–Hilbert boundary value problem, with an arbitrary index, for both homogeneous and inhomogeneous Cauchy–Riemann equations on an isosceles orthogonal triangle in the complex plane.

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In general, an explicit solvability of the Riemann–Hilbert problem on the triangle domain Δ cannot be directly obtained by the conformal invariance with the unit disc or the half-plane, using the Schwarz–Christoffel formula. Therefore, to handle the Riemann–Hilbert boundary value problem explicitly we use the technique of plane parqueting, which is used in [19,20,22,39]. This function analytic method is one of the remarkable ingredients in this work. Another important result is related to a class of integral operators $S_{\alpha,\kappa}$, $T_{\alpha,\kappa}$, with explicit kernel. Since such operators are important in studying the boundary value problems for complex partial differential equations and the singular integral equations, see, e.g. [28], we have obtained some important properties of them.

The article is organized as follows: In Section 2, a review of the Schwarz–Poisson formula for an isosceles orthogonal triangle and an explicit solvability of the Schwarz problem in the triangle will be given. Basic properties of related integral operators are recalled. The Riemann–Hilbert problems of non-negative index are investigated in Section 3. Firstly, the boundary behaviors of a Schwarz-type operator and a Pompeiu-type operator are investigated. Section 4 is devoted to study the Riemann–Hilbert problems of negative index.

2. Preliminaries

Let Δ be the isosceles orthogonal triangle with the vertices 0, 1, i. The boundary $\partial\Delta$ of the domain Δ consists of three sides, oriented counter-clockwise and are denoted by $[1,i]$, $[i,0]$, $[0,1]$, i.e. $\partial\Delta = [1,i] \cup [i,0] \cup [0,1]$.

By the technique of plane parqueting [3,21,39], the complex plane $\mathbb{C}$ can be divided into infinitely many triangles, which are congruent to the triangle Δ. Firstly, by reflecting Δ to the right repeatedly, one gets the symmetric points

$$\begin{aligned} z_1 &= 1 - i(\bar{z} - 1) = -i\bar{z} + 1 + i, \\ z_2 &= 1 - i(z - 1) = -iz + 1 + i, \\ z_3 &= 2 - \bar{z}, \\ z_4 &= 2 + z, \end{aligned} \tag{2.1}$$

for $z \in \Delta$. It is clear that the reflection along the horizontal direction possesses the minimum positive period 2. By the reflection along the vertical direction, the basic period $2i$ is similarly obtained. By the reflection on the real axis, the corresponding symmetric points of z, z_1, z_2, z_3 are denoted as $\bar{z}$, $\overline{z_1}$, $\overline{z_2}$, $\overline{z_3}$, respectively.

Let

$$\Omega_{m,n} = 2m + 2ni, \quad m, n \in \mathbb{Z}, \tag{2.2}$$

all the reflection points of $z \in \Delta$ are

$$\begin{aligned} &z + \Omega_{m,n}, \quad z_1 + \Omega_{m,n}, \quad z_2 + \Omega_{m,n}, \quad z_3 + \Omega_{m,n}, \\ &\bar{z} + \Omega_{m,n}, \quad \overline{z_1} + \Omega_{m,n}, \quad \overline{z_2} + \Omega_{m,n}, \quad \overline{z_3} + \Omega_{m,n} \end{aligned} \tag{2.3}$$

where z_1, z_2, z_3 are defined by (2.1) and $\Omega_{m,n}$ is defined by (2.2).

Applying the Cauchy–Pompeiu formula [13,37], which is still valid for the triangle domain, to both inside and outside the triangle Δ, leads to the following Schwarz–Poisson-type formula for the triangle domain Δ.

Theorem 2.1. *(See [39].) If $w \in C^1(\Delta;\mathbb{C}) \cap C(\overline{\Delta};\mathbb{C})$, then*

$$w(z) = w(\alpha) + \frac{1}{\pi i} \int_{\partial\Delta} \operatorname{Re} w(\zeta) \sum_{m,n} \big[g_{m,n}(\zeta, z) - g_{m,n}(\zeta, \alpha)\big] d\zeta$$