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The equivalent refraction index for the acoustic scattering by many small obstacles: With error estimates

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A B S T R A C T

Let M be the number of bounded and Lipschitz regular obstacles D_j, $j := 1, ..., M$ having a maximum radius a, $a \ll 1$, located in a bounded domain Ω of $\mathbb{R}^3$. We are concerned with the acoustic scattering problem with a very large number of obstacles, as $M := M(a) := O(a^{-1})$, $a \to 0$, when they are arbitrarily distributed in Ω with a minimum distance between them of the order $d := d(a) := O(a^t)$ with t in an appropriate range. We show that the acoustic farfields corresponding to the scattered waves by this collection of obstacles, taken to be soft obstacles, converge uniformly in terms of the incident as well as the propagation directions, to the one corresponding to an acoustic refraction index as $a \to 0$. This refraction index is given as a product of two coefficients $\mathbf{C}$ and K, where the first one is related to the geometry of the obstacles (precisely their capacitance) and the second one is related to the local distribution of these obstacles. In addition, we provide explicit error estimates, in terms of a, in the case when the obstacles are locally the same (i.e. have the same capacitance, or the coefficient $\mathbf{C}$ is piecewise constant) in Ω and the coefficient K is Hölder continuous. These approximations can be applied, in particular, to the theory of acoustic materials for the design of refraction indices by perforation using either the geometry of the holes, i.e. the coefficient $\mathbf{C}$, or their local distribution in a given domain Ω, i.e. the coefficient K.

1. Introduction and statement of the results

Let $B_1, B_2, \ldots, B_M$ be M open, bounded and simply connected sets in $\mathbb{R}^3$ with Lipschitz boundaries containing the origin. We assume that the Lipschitz constants of B_j, $j = 1, ..., M$ are uniformly bounded. We set $D_m := \epsilon B_m + z_m$ to be the small bodies characterized by the parameter $\epsilon > 0$ and the locations

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$z_m \in \mathbb{R}^3$, $m = 1, \dots, M$. We denote by U^s the acoustic field scattered by the M small and soft bodies $D_m \subset \mathbb{R}^3$ due to the incident plane wave $U^i(x, \theta) := e^{ikx\cdot\theta}$, with the incident direction $\theta \in \mathbb{S}^2$, with $\mathbb{S}^2$ being the unit sphere. Hence the total field $U^t := U^i + U^s$ satisfies the following exterior Dirichlet problem of the acoustic waves

$$(\Delta + \kappa^2)U^t = 0 \quad \text{in } \mathbb{R}^3 \setminus \left(\bigcup_{m=1}^{M} \overline{D}_m \right), \tag{1.1}$$

$$U^t\big|_{\partial D_m} = 0, \quad 1 \le m \le M, \tag{1.2}$$

$$\frac{\partial U^s}{\partial |x|} - i\kappa U^s = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty \quad \text{(S.R.C)} \tag{1.3}$$

where $\kappa > 0$ is the wave number, $\kappa = 2\pi/\lambda$, λ is the wave length and S.R.C stands for the Sommerfield radiation condition. The scattering problem (1.1)–(1.3) is well posed in appropriate spaces, see [5,10] for instance, and the scattered field $U^s(x, \theta)$ has the following asymptotic expansion:

$$U^s(x, \theta) = \frac{e^{i\kappa|x|}}{|x|} U^\infty(\hat{x}, \theta) + O\left(|x|^{-2}\right), \quad |x| \to \infty, \tag{1.4}$$

with $\hat{x} := \frac{x}{|x|}$, where the function $U^\infty(\hat{x}, \theta)$ for $(\hat{x}, \theta) \in \mathbb{S}^2 \times \mathbb{S}^2$ is called the far-field pattern. We recall that the fundamental solution $\Phi_\kappa(x, y)$ of the Helmholtz equation in $\mathbb{R}^3$ with the fixed wave number κ is given by $\Phi_\kappa(x, y) := \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$, for all $x, y \in \mathbb{R}^3$.

Definition 1.1. We define

1. a as the maximum among the diameters, *diam*, of the small bodies D_m, i.e.

$$a := \max_{1 \le m \le M} diam(D_m) \left[= \epsilon \max_{1 \le m \le M} diam(B_m)\right], \tag{1.5}$$

2. d as the minimum distance between the small bodies $\{D_1, D_2, \dots, D_m\}$, i.e.

$$d := \min_{\substack{m \ne j \\ 1 \le m, j \le M}} d_{mj}, \tag{1.6}$$

where $d_{mj} := dist(D_m, D_j)$. We assume that

$$0 < d \le d_{\max}, \tag{1.7}$$

and $d_{\max}$ is given.

3. $\kappa_{\max}$ as the upper bound of the used wave numbers, i.e. $\kappa \in [0, \kappa_{\max}]$.

We assume that $D_m = \epsilon B_m + z_m$, $m = 1, \dots, M$, with the same diameter a, are non-flat Lipschitz obstacles, i.e. D_m's are Lipschitz obstacles and there exist constants $t_m \in (0, 1]$ such that

$$B^3_{t_m \frac{a}{2}}(z_m) \subset D_m \subset B^3_{\frac{a}{2}}(z_m), \tag{1.8}$$

where t_m are assumed to be uniformly bounded from below by a positive constant.

In a recent work [2], we have shown that there exist two positive constants a_0 and c_0 depending only on the Lipschitz character of B_m, $m = 1, \dots, M$, $d_{\max}$ and $\kappa_{\max}$ such that if