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Two new regularity criteria for nematic liquid crystal flows

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ABSTRACT

In this study, we present two new regularity criteria for three-dimensional nematic liquid crystal flows. More precisely, we show that if $\partial_3 u \in L^\beta(0,T;L^\alpha(R^3))$ with $\frac{2}{\beta}+\frac{3}{\alpha}\le\frac{3(\alpha+2)}{4\alpha}$ $(\alpha>2)$, or u_3, $\nabla d \in L^\beta(0,T;L^\alpha(R^3))$ with $\frac{2}{\beta}+\frac{3}{\alpha}\le\frac{3}{4}+\frac{1}{2\alpha}$ $(\alpha>\frac{10}{3})$, then the corresponding weak solution (u,d) can be extended beyond T.

1. Introduction

In this study, we present two new regularity criteria for the three-dimensional (3D) nematic liquid crystal flows:

$$\begin{cases} u_t+(u\cdot\nabla)u+\nabla p-\nu\Delta u=-\lambda\nabla\cdot(\nabla d\otimes\nabla d), & x\in R^3,\ t>0\\ d_t+(u\cdot\nabla)d=\gamma\big(\Delta d-f(d)\big), & x\in R^3,\ t>0\\ \nabla\cdot u=0, & x\in R^3,\ t>0\end{cases}\tag{1.1}$$

with initial data

$$(u,d)|_{t=0}=(u_0,d_0),\quad x\in R^3,\tag{1.2}$$

where $u(x,t)$ is the velocity field, $d(x,t)$ represents the macroscopic average of the nematic liquid crystal orientation field, and $p(x,t)$ is the scalar pressure. The symbol $\nabla d\otimes\nabla d$ denotes a matrix where the (i,j)th entry is given by $\partial_i d\cdot\partial_j d$ for $1\le i,j\le 3$, and where $f(d)=\frac{1}{\varepsilon^2}(|d|^2-1)d$. $\nu,\lambda,\gamma,\varepsilon$ are positive constants, which we assume are all one, for simplicity.

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The hydrodynamic theory of liquid crystals was established by Ericksen and Leslie [5–7,16], and model (1.1) is a simplified version of the Ericksen–Leslie model, which was first introduced by Lin in [17]. One of the most significant studies in this area was made by Lin and Liu [19], where they established a global existence theorem for weak solutions and classical solutions. Fan and Guo [8] proved the following regularity criteria: $u \in L^s(0,T;\dot{M}_{p,q}(R^3))$ with $\frac{2}{s}+\frac{3}{p}=1$, $p>3$, $p \geq q \geq 1$, or $\nabla u \in L^s(0,T;\dot{M}_{p,q}(R^3))$ with $\frac{2}{s}+\frac{3}{p}=2$, $p>\frac{3}{2}$, $p \geq q \geq 1$. Liu, Zhao and Cui [22] established the regularity criterion for (1.1) as follows:

$$\int_0^T \|\partial_3 u(\tau)\|_{L^\alpha}^\beta d\tau < \infty, \quad \text{with} \quad \frac{2}{\beta}+\frac{3}{\alpha} \leq 1, \quad \alpha > 3. \tag{1.3}$$

Several other interesting and important regularity criteria for nematic liquid crystal flows have also been investigated (e.g., [10,12–14,18,20,21,24,27] and the references therein), some of which were motivated by studies based on the Navier–Stokes equations. However, the regularity of solutions in terms of one component of the velocity field to the system is still an open issue. When the director field $d \equiv 1$, the system (1.1) becomes the well-known Navier–Stokes equations. The regularity of solutions to the 3D Navier–Stokes equations has been investigated widely during the past 50 years (e.g., [2–4,9,15,23,25,26,28]). Recently, Cao and Titi [2] proved the regularity of weak solutions to the three-dimensional Navier–Stokes equations subject to periodic boundary conditions or in the whole space in terms of one component of the velocity field. Specifically, they gave the following condition:

$$\int_0^T \|u_3(\tau)\|_{L^\alpha}^\beta d\tau < \infty, \quad \text{with} \quad \frac{2}{\beta}+\frac{3}{\alpha} \leq \frac{2(\alpha+1)}{3\alpha}, \quad \alpha > \frac{7}{2}. \tag{1.4}$$

Based on this method, Zhou and Pokorný [28] further provided the following sufficient condition:

$$\int_0^T \|u_3(\tau)\|_{L^\alpha}^\beta d\tau < \infty, \quad \text{with} \quad \frac{2}{\beta}+\frac{3}{\alpha} \leq \frac{3}{4}+\frac{1}{2\alpha}, \quad \alpha \in \left(\frac{10}{3},\infty\right]. \tag{1.5}$$

Motivated by previous research, one of the aims of the present study was to establish a regularity criterion for 3D nematic liquid crystal flows in terms of one directional derivative of the velocity, while another was to establish a regularity criterion by providing a sufficient condition in terms of one velocity component and the nematic liquid crystal orientation field.

Theorem 1.1. *Let $(u_0,d_0) \in H^1(R^3) \times H^2(R^3)$ with the initial data $\operatorname{div} u_0 = 0$, and let the pair (u,b) be the weak solution to the liquid crystal flows (1.1)–(1.2) on $[0,T)$ for some $0<T<\infty$. If u satisfies*

$$\partial_3 u \in L^\beta\big(0,T;L^\alpha(R^3)\big) \quad \text{with} \quad \frac{3}{\alpha}+\frac{2}{\beta} \leq \frac{3(\alpha+2)}{4\alpha}, \quad \alpha > 2. \tag{1.6}$$

Then, (u,d) can be extended beyond T.

Theorem 1.2. *Let $(u_0,d_0) \in H^1(R^3) \times H^2(R^3)$ with the initial data $\operatorname{div} u_0 = 0$, and let the pair (u,b) be the weak solution to the liquid crystal flows (1.1)–(1.2) on $[0,T)$ for some $0<T<\infty$. If u satisfies*

$$u_3, \nabla d \in L^\beta\big(0,T;L^\alpha(R^3)\big), \quad \text{with} \quad \frac{2}{\beta}+\frac{3}{\alpha} \leq \frac{3}{4}+\frac{1}{2\alpha}, \quad \alpha > \frac{10}{3}. \tag{1.7}$$

Then, (u,d) can be extended beyond T.