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Remarks on unboundedness of set-valued Itô stochastic integrals

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ABSTRACT

The paper deals with set-valued stochastic integrals which are defined as a random multifunctions. Integrable boundedness of a such integral is one of the most important features in potential applications. Unfortunately, up to now there were no correct proofs of such property. Surprisingly, also a negative answer to this problem is still not explained correctly. Hence the problem seems to be still undetermined. We shall show that in general the answer is negative. We shall provide several relatively simple examples both in convex and nonconvex-valued case. As a consequence, we will show that under some restrictions on the class of selections of the integrand the integrable boundedness of set-valued Itô's integral is equivalent to its single valuedness.

1. Introduction

For recent twenty years notions of set-valued Itô stochastic integrals for multivalued functions have attracted the interest of many authors both from theoretical and practical points of view. They are the main tools used in the theory of stochastic differential inclusions, theory of set-valued stochastic differential equations and fuzzy stochastic differential equations (see e.g. [2–5,8,12,15,16,18–43] and references therein). The first formulation was proposed by G. Bocşan in [8]. Unfortunately, the definition and some properties of such an integral were not quite correct. Different definitions of the set-valued Itô's integrals have been independently formulated by F. Hiai in [12] and M. Kisielewicz in [19]. According to [12] and [19] the set-valued Itô's integral (driven by a Brownian motion) is understood as subset of the space of all square integrable random variables with values in R^d (or more generally, with values in a Banach space). Such an approach reflects R. Aumann's idea of a set-valued integral in a deterministic case [7]. In fact such a subset is defined as a family of Itô's stochastic integrals taken over the set of Itô's integrable selections of a given nonanticipating and integrally bounded multivalued mapping. In what follows such defined integral is called nowadays a functional set-valued stochastic Itô's integral [24]. For properties and wide range of applications

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of the functional set-valued Itô's integral we refer the reader to the monograph [25] and references therein. At that time, one of the main question was if there existed a set-valued random variable, called the set-valued Itô's integral, for which the set of its all square integrable selections coincides with a functional set-valued Itô's integral for a given multivalued function. According to [13, Theorem 2.38] this issue is equivalent to the question about the decomposability property for the functional set-valued Itô's integral as a subset in L^2. For several years it was an open problem. A negative answer to this question was given for the first time (according to my knowledge) in 2011 in [36] where a simple counterexample was presented. A year later, in [24] it was shown that the functional set-valued Itô's integral is decomposable if and only if it is a singleton, which means that the integrand has to be single-valued as well. Hence it was evident that the only natural way of defining the set-valued Itô's integral (as a random multifunction) is to consider the closed and decomposable hull of a functional set-valued Itô's stochastic integral as the set of its all square integrable selections. Such an approach was proposed for the first time in 2003 by E.J. Jung and J.H. Kim in [16] (of course without any explanation because the justification of this approach was an open problem at that time). Unfortunately, some properties of the set-valued Itô's integral presented in [16] and the proofs of some theorems are not correct (see [24] for detailed remarks). One of them it is a property of integrable boundedness of such integrals for integrally bounded multivalued integrands. This property seems to be crucial both in the theory and potential applications of a set-valued Itô's integration. In particular the negative answer to this problem would have several serious consequences. For instance, the notions based on a such integral like stochastic inclusions, set-valued stochastic equations and fuzzy-valued stochastic equations could not be formulated properly unless the diffusion term is single-valued. In 2007 Y. Ogura presented a simple example (see Theorem 1 in [41]) of a bounded deterministic multivalued function for which (according to this author) the set-valued Itô's integral is unbounded. The idea was to construct a sequence of selection processes which stochastic integrals diverge a.e. Unfortunately, also in this case the argumentation was not correct. It follows from the fact that constructed selection processes are not nonanticipating so they are not integrable in Itô's sense. Thus, it seems that the question still unanswered is whether the set-valued stochastic Itô's integral may be unbounded for an integrally bounded multivalued integrand. Equivalently it is a question of boundedness (or unboundedness) of the set of all L^2-selections of the set-valued Itô's stochastic integral. The aim of this work is to determine the problem. In the paper we shall show that the set-valued Itô's integral may be unbounded for integrally bounded or even bounded multivalued integrands. We shall provide several relative simple examples both in convex and nonconvex case. As a consequence, we will show that under some restrictions on the class of selections of the integrand the integrable boundedness of set-valued Itô's integral is equivalent to its single valuedness. Surprisingly, we shall only use some basic facts from set-valued analysis and from the theory of Gaussian processes. The main idea is to apply Sudakov–Slepian–Fernique and Fernique inequalities [1,9] for our aims.

2. Basic notions and auxiliary results

For the convenience of the reader we repeat the relevant material from [1,9,11,13,14], thus making our exposition more self contained. We shall start with some notions and facts from set-valued analysis. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a separable Banach space and let $(U, \mathcal{U}, \mu)$ be a complete finite measure space. By $L^p(U, \mathcal{U}, \mu; \mathcal{X})$, $p \geq 1$ we denote L^p-Bochner integrable functions. Let M be a set of $\mathcal{U}$-measurable mappings $f : U \to \mathcal{X}$. The set M is said to be $\mathcal{U}$-decomposable if for every $u_1, u_2 \in M$ and every $A \in \mathcal{U}$ it holds $u_1 \mathbb{I}_A + u_2 \mathbb{I}_{U \setminus A} \in M$. For a nonempty set $\Gamma \subset L^p(U, \mathcal{U}, \mu; \mathcal{X})$ by $\mathrm{dec}(\Gamma)$ and $\mathrm{cl}_{L^p}\mathrm{dec}(\Gamma)$ we denote the smallest decomposable and the smallest closed and decomposable set containing Γ. They are called, respectively, decomposable hull and closed decomposable hull for Γ (see [11] for details). Let $N \geq 1$. By $\Pi_N(U, \mathcal{U})$ we denote the set of all $\mathcal{U}$-measurable partitions $(A_i)_{i=1}^N$ of the space U. Then for every finite subset $\Gamma := \{u_1, u_2, ..., u_N\} \subset L^p(U, \mathcal{U}, \mu; \mathcal{X})$ it holds