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Generic profit singularities in time averaged optimization for phase transitions in polydynamical systems

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ABSTRACT

We consider the optimization problem that consists in maximizing the time averaged profit for a motion of a smooth polydynamical system on the circle in the presence of a smooth profit density. When the problem depends on a k-dimensional parameter the optimal averaged profit is a function of the parameter. It is known that an optimal motion can always be selected among stationary strategies and a special type of periodic cyclic motions called *level cycles*. We present the classification of all generic singularities of the optimal averaged profit when $k \leq 2$ for phase transitions between these two optimal strategies.

1. Introduction

Consider a *smooth control system* on the circle S^1:

$$\dot{x} = v(x, u)$$

where x is an angle on the circle and u is a control parameter that belongs to the control space U, which is a smooth closed manifold (or a disjoint union of smooth closed manifolds) with at least two different points.

Each vector field $v(\cdot, u)$ that is obtained by fixing the control parameter value u is called an *admissible velocity*. Given a point x_0 on the circle, the *set of admissible velocities at* x_0 is given by

$$V(x_0) = \left\{v(x_0, u)\colon\ u \in U\right\}.$$

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A motion $x : \mathbb{R} \to S^1$ of the control system is said to be an *admissible motion* if it is absolutely continuous and the velocity of motion at each time of differentiability, $\dot{x}(t)$, is an admissible velocity, that is, $\dot{x}(t) \in V(x(t))$.

Remark 1. The phase space is compact and so any admissible motion of the control system can be defined for all $t \in \mathbb{R}$.

In this context, the existence of a smooth *profit density* $f : S^1 \to \mathbb{R}$ on the circle leads to the following optimal control problem:

To maximize the averaged profit on the infinite time horizon

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f\big(x(t)\big)dt$$

over all the system's admissible motions on the positive semiaxis.

We take the upper limit whenever the above limit does not exist.

This problem of control theory includes in particular the optimization of the averaged profit of periodic processes when the phase space is the circle. References to various applications of periodic control problems (e.g. economic, electrical engineering and chemical reaction engineering) as well as a case study to production planning can be found in [7]. In this paper we approach this problem from the singularity theory point of view.

Consider that this optimization problem depends on a parameter p belonging to a smooth manifold, that is, that both the control system and the profit density depend on a parameter p. Then, the optimal strategy can vary with p and the optimal averaged profit on the infinite time horizon is the function A defined by

$$A(p) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\big(x(t), p\big)dt,$$

and can have singularities (points where it is not smooth). This brings us to the problem of classifying such singularities.

This problem was firstly considered in [1] and then in [3] and [4]. In these works, the determination of the optimal averaged profit on the infinite horizon of a controlled dynamical system is based on two different types of admissible motions: a *level cycle* (motion that uses the maximum, respectively minimum, velocity when the profit density is less, respectively greater, than a specific constant) and a *stationary strategy* (motion associated with an equilibrium point of the controlled dynamical system, i.e., a point where the convex hull of all admissible velocities contains the zero velocity).

In fact, the maximal averaged profit can always be provided by an admissible motion of these types (see [4]). Therefore in order to classify the singularities of the optimal averaged profit, it is enough to consider the following three situations: singularities for (a) stationary strategies, (b) level cycles and (c) transitions between stationary strategies and level cycles.

The classification of all generic singularities was already treated in the case of a one dimensional parameter ([1,3,4] for a control space without boundary and [8] for a control space with a regular boundary).

In this work we consider this Singularity Theory problem for a special type of control systems, namely, for *polydynamical systems*, that is, for control systems with a finite number of (at least two) admissible velocities. Note that in this case the control space U is of the form $\{1, \ldots, n\}$, $n \geq 2$ and, for simplicity, we denote by v_i the admissible velocity $v(\cdot, \cdot, i)$, $1 \leq 1 \leq n$. Hence, the set of admissible velocities at a point (x, p) takes the form