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Plane stress problems in nonlocal elasticity: finite element solutions with a strain-difference-based formulation

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ABSTRACT

An enhanced computational version of the finite element method in the context of nonlocal strain-integral elasticity of Eringen-type is discussed. The theoretical bases of the method are illustrated focusing the attention on numerical and computational aspects as well as on the construction of the nonlocal elements matrices. Two numerical examples of plane stress nonlocal elasticity are presented to show the potentials and the limits of the promoted approach.

1. Problem position, motivations and goals of the present study

Constitutive material models established in the field of solid mechanics for classical continuous media cannot describe problems where micro- and nano-effects play a crucial role in the mechanical behavior. Typical examples, among many others, are the singular stress field predicted at a sharp crack-tip in a continuum fracture mechanics problem (see e.g. [14]), or the inability of classical continuum mechanics theory in describing deformation phenomena of nanotubes or other nanoscale structures (see e.g. [21]), or, also, wave dispersion, strain softening, concomitant size effects (see e.g. [8]). The simplest approaches to overcome such inherent limitations of classical theories are the so-called nonlocal continuum approaches based on an *enrichment* of the classical modeling by keeping the hypothesis of *continuity* but introducing an *internal length material scale* able to take into account the phenomena imputable to the micro- or nano-structure. There are several ways to act in this direction as witnessed by the broad relevant literature.

The *gradient approach*, promoted by Aifantis in the 1980s in a simplified version (with only three constants, including the Lamé ones) of previous gradient formulations that can be traced back to the sixties (see e.g. [3,6] and the references therein), as well as the *integral approach*, proposed by Eringen [12,13], are the most widely used. *Peridynamic models*, first conceived by Silling [33], or *continualization procedures*

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(see e.g. [5]) are also well established and very promising approaches. The remarkable works [32,9,8,19], for conditions ensuring the existence of fundamental solutions and for nonlocal models of integral-type framed in the broader context of plasticity and damage, can be also referenced. Nonlocal variational formulations and unifying thermodynamically consistent treatments can be finally found in the landmark papers of Polizzotto [26–29]. The list of relevant contributions, without any pretense of completeness, is only meant to fix the background of the present study which promotes, from a computational point of view, a *nonlocal elasticity of integral type* carried on by the so-called *nonlocal finite element method* (NL-FEM).

The theoretical bases of what is discussed further down have been given in [26,30,31], while the first tentatives of computational nature are those given in [24,25]. In the former three theoretical papers the NL-FEM was conceived together with an Eringen-type nonlocal integral elasticity model in which the stress is expressed as the sum of two contributions: one is the standard local stress and the other, of nonlocal nature, is given in terms of an *averaged strain difference field*. In the latter couple of papers the NL-FEM was implemented and tested with reference to simple, but effective, examples also dealing with nonhomogeneous 2D nonlocal problems.

On the base of such previous studies, the main goals of the present study are: i) to eliminate some computational drawbacks of the previous NL-FEM formulation due to the excessive dimensions of the nonlocal operators related to the ones of the whole analyzed structure; ii) to tackle a benchmark problem solved by other Researchers with different, or alternative, theoretical as well as numerical, approaches; iii) to point out potentialities and limits of the promoted NL-FEM having also a look on possible improvements and future field of application.

The plan of the paper is the following. After this introductory section, in Section 2 the strain-difference based nonlocal model and the NL-FEM theoretical developments are given in an abridged form. Section 3 goes into the details of the numerical implementation, explaining how to build the requested nonlocal operators giving also a flow-chart of the numerical procedure. Section 4 is devoted both to the numerical results obtained for a couple of problems and to a critical investigation on the effects of some material parameters. Concluding remarks are finally given in Section 5 which closes the paper.

Notation. A compact notation is used throughout, with bold-face letters for vectors and tensors. The "dot" and "colon" products between vectors and tensors denote simple and double index contraction operations, respectively. For instance: $\boldsymbol{u}\cdot\boldsymbol{v}=u_i v_i$, $\boldsymbol{\sigma}:\boldsymbol{\varepsilon}=\sigma_{ij}\,\varepsilon_{ij}$, $\boldsymbol{\sigma}\cdot\boldsymbol{n}=\{\sigma_{ij}\,n_j\}$, $\boldsymbol{D}:\boldsymbol{\varepsilon}=\{D_{ijhk}\,\varepsilon_{hk}\}$. The symbol $:=$ means equality by definition. Other symbols will be defined in the text where they appear for the first time.

2. Eringen-type nonlocal integral elasticity: strain-difference-based model and NL-FEM formulation

2.1. The strain-difference-based nonlocal model

Let us consider a nonlocal linear elastic material occupying, in its undeformed state, a three-dimensional Euclidean domain of volume V referred to orthogonal Cartesian coordinates $\boldsymbol{x}=(x_1,x_2,x_3)$. The nonlocal feature of the material is expressed by the following constitutive relation:

$$\boldsymbol{\sigma}(\boldsymbol{x})=\boldsymbol{D}(\boldsymbol{x}):\boldsymbol{\varepsilon}(\boldsymbol{x})-\alpha\int_V \boldsymbol{\mathcal{J}}(\boldsymbol{x},\boldsymbol{x}'):[\boldsymbol{\varepsilon}(\boldsymbol{x}')-\boldsymbol{\varepsilon}(\boldsymbol{x})]\,\mathrm{d}V' \quad \forall\,(\boldsymbol{x},\boldsymbol{x}')\in V. \tag{1}$$

Equation (1), proposed by Polizzotto et al. [31], simply states that the stress response, $\boldsymbol{\sigma}(\boldsymbol{x})$, to a given strain field, $\boldsymbol{\varepsilon}(\boldsymbol{x})$, is the sum of two contributions. The first one, of *local nature*, is governed by the standard symmetric and positive definite elastic moduli tensor $\boldsymbol{D}(\boldsymbol{x})$ assumed variable in space so that, as in the quoted paper, nonhomogeneous materials can, if necessary, be considered. The second one, of *nonlocal nature*, depends on the strain difference field $[\boldsymbol{\varepsilon}(\boldsymbol{x}')-\boldsymbol{\varepsilon}(\boldsymbol{x})]$ through the symmetric nonlocal tensor $\boldsymbol{\mathcal{J}}(\boldsymbol{x},\boldsymbol{x}')$